PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

Plan for Lecture 37

Review of Mathematical Techniques – Green's function methods

1. Methods for two and higher dimensions
2. Examples
3. Course evaluation forms

29	Fri, 11/06/2015	Chap. 10	Surface waves on fluids	#27
30	Mon, 11/09/2015	Chap. 10	Surface waves on fluids	#28
31	Wed, 11/11/2015	Chap. 11	Heat Conduction	#29
32	Fri, 11/13/2015	Chap. 12	Viscosity	#30
33	Mon, 11/16/2015	Chap. 12	Viscosity	Prepare presentation.
34	Wed, 11/18/2015	Chap. 12	Viscosity	Prepare presentation.
35	Fri, 11/20/2015	Chap. 12	Elastic Continua	Prepare presentation.
36	Mon, 11/23/2015	Chap. 13	Review of Mathematical Methods	Prepare presentation.
	Wed, 11/25/2015		*Thanksgiving Holiday*	
	Fri, 11/27/2015		*Thanksgiving Holiday*	
37	Mon, 11/30/2015	Chap. 13	Review of Mathematical Methods	Prepare presentation.
	Wed, 12/02/2015		Student presentations I	
	Fri, 12/04/2015		Student presentations II	
	Mon, 12/07/2015		Begin Take-home final	

**Note: The final exam has the take-home form similar to that of the mid-term. In order to accommodate your schedules, it will be available on 12/04 and will be due before 9 AM on 12/14.

PHY 711 Presentation Schedule

Wednesday, December 2, 2015

	Presenter	Title of presentation
10:00-10:17	Taylor Ordines	Scattering Cross Section of $\psi^*\psi\phi$ Theory
10:17-10:34	Peiyun Li	Catenary - Physical Thinking on Hanging Rope
10:34-10:51	Gabriel Marcus	A Study of Nonlinear Dynamics and Chaos

Friday, December 4, 2015

	Presenter	Title of presentation
10:00-10:17	Eric Henderson	Numerical Evaluation of the Scattering Cross Section
10:17-10:34	Andrew Zeidell	Modeling Viscous Fluid Flow Through an Atomizer
10:34-10:51	Tong Ren	Modelling of Symmetric Top with Fluid Inside: Ideal and Viscous

Some guidelines about the presentations

1. In your presentations, please make sure to acknowledge all of your sources.
2. We have allotted 17 minutes including questions for each presentation.
3. In order to encourage participation, points will be awarded for questions from the audience.
4. For efficiency, you may wish to email me your talk an use my computer for the presentation.
5. At the end each session, please email me your presentations and any supplementary materials.

Green's function solution methods for one-dimensional second order (Sturm-Liouville) differential equations:

Differential equation to be solved for $\varphi(x)$:

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)\varphi(x)=F(x)$$

Green's function :

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)G_\lambda(x,x')=\delta(x-x')$$

Formal solution to differential equation:

$$\varphi_\lambda(x)=\varphi_{\lambda 0}(x)+\int G_\lambda(x,x')F(x')dx'$$

Solution to problem with *F(x)=0*

Green's function construction method #1 -- using homogeneous solutions to Sturm-Liouville equations

Find two independent solution to the homogeneous equation:

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)g_i^\lambda(x)=0 \qquad \text{for} \quad i=a,b$$

Let

$$G_\lambda(x,x')=\frac{1}{W}g_a^\lambda(x_<)g_b^\lambda(x_>)$$

Where: $W\equiv-\tau(x)\left(g_a^\lambda(x)\frac{d}{dx}g_b^\lambda(x)-g_b^\lambda(x)\frac{d}{dx}g_a^\lambda(x)\right)$

Green's function construction method #2 -- using eigenfunctions ot the Sturm-Liouville equations

Note that in general there is a finite domain: $x_l \le x \le x_u$

Eigenfunctions:

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)\right)f_n(x)=\lambda_n\sigma(x)f_n(x)$$

Properties of eigenfunctions:

Orthogonality: $\int \sigma(x)f_n(x)f_m(x)dx=\delta_{nm}N_n,$

where $N_n \equiv \int \sigma(x)(f_n(x))^2dx.$

Completeness: $\sigma(x)\sum_n \frac{f_n(x)f_n(x')}{N_n}=\delta(x-x')$

Let $G_\lambda(x,x')=\sum_n \frac{f_n(x)f_n(x')/N_n}{\lambda_n-\lambda}$

Green's function solution methods for one-dimensional second order (Sturm-Liouville) differential equations -- comparison of method #1 and #2

Method #1: $G_\lambda(x,x')=\frac{1}{W}g_a^\lambda(x_<)g_b^\lambda(x_>)$

- Single discontinuous integral for evaluating full solution
- Only works in one dimension

Method #2: $G_\lambda(x,x')=\sum_n \frac{f_n(x)f_n(x')/N_n}{\lambda_n-\lambda}$

- Infinite series of continuous integrals for evaluating full solution
- Can be extended to multiple dimensions

Green's function methods in multiple dimensions --

Example system – Poisson equation

$$\nabla^2\Phi(\mathbf{r})=-\frac{\rho(\mathbf{r})}{\varepsilon_0}$$

Charge density

Electrostatic potential

Solution to Poisson equation using Green's function $G(\mathbf{r},\mathbf{r}')$:

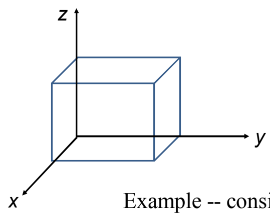
$$\Phi(\mathbf{r})=\frac{1}{4\pi\varepsilon_0}\int_V d^3r'\rho(\mathbf{r}')G(\mathbf{r},\mathbf{r}')+$$

$$\frac{1}{4\pi}\int_S d^2r'\left[G(\mathbf{r},\mathbf{r}')\nabla'\Phi(\mathbf{r}')-\Phi(\mathbf{r}')\nabla'G(\mathbf{r},\mathbf{r}')\right]\cdot\hat{\mathbf{r}}'.$$

Boundary value terms

Green's function equation in this case:

$$\nabla^2 G(\mathbf{r},\mathbf{r}') = -4\pi\delta(\mathbf{r}-\mathbf{r}')$$



Example -- consider cartesian coordinates:

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Green's function using method #2 in all three dimensions

Let $\{u_n(x)\}$, $\{v_n(y)\}$, $\{w_n(z)\}$ denote complete orthogonal function sets in the x, y, and z dimensions, respectively. The Green's function construction becomes:

$$G(x,x',y,y',z,z') = 4\pi \sum_{lmn} \frac{u_l(x)u_l(x')v_m(y)v_m(y')w_n(z)w_n(z')}{\alpha_l + \beta_m + \gamma_n},$$

where

$$\frac{d^2}{dx^2}u_l(x) = -\alpha_l u_l(x), \ \frac{d^2}{dy^2}v_m(y) = -\beta_m v_m(y), \text{ and } \frac{d^2}{dz^2}w_n(z) = -\gamma_n w_n(z).$$

Green's function using method #2 in all two dimensions and method #1 in the third dimension

Let $\{u_n(x)\}$, $\{v_n(y)\}$ denote complete orthogonal function sets in the x, and y dimensions, respectively. The Green's function construction becomes:

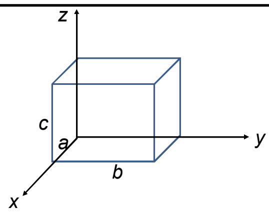
$$G(x,x',y,y',z,z') = 4\pi \sum_{lm} u_l(x)u_l(x')v_m(y)v_m(y')g_{lm}(z,z'),$$

where

$$\frac{d^2}{dx^2}u_l(x) = -\alpha_l u_l(x), \ \frac{d^2}{dy^2}v_m(y) = -\beta_m v_m(y),$$

$$\text{and } \left(\frac{d^2}{dz^2} - \alpha_l - \beta_m\right) g_{lm}(z,z') = -4\pi\delta(z-z')$$

Example

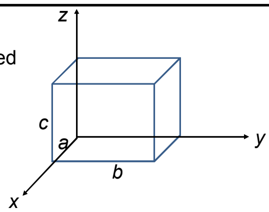


Suppose the boundary conditions require the Green's function to vanish on all 6 faces: $x=0, x=a, y=0, y=b, z=0, z=c$

Method #2 construction:

$$G(x,x',y,y',z,z') = 4\pi \frac{8}{abc}\sum_{lmn} \frac{\sin\left(\frac{l\pi x}{a}\right)\sin\left(\frac{l\pi x'}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{m\pi y'}{b}\right)\sin\left(\frac{n\pi z}{c}\right)\sin\left(\frac{n\pi z'}{c}\right)}{\left(\left(\frac{l\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2+\left(\frac{n\pi}{c}\right)^2\right)}$$

Example continued



Suppose the boundary conditions require the Green's function to vanish on all 6 faces: $x=0, x=a, y=0, y=b, z=0, z=c$

Combined construction:

$$G(x,x',y,y',z,z') =$$

$$-4\pi \frac{4}{ab}\sum_{lm} \frac{\sin\left(\frac{l\pi x}{a}\right)\sin\left(\frac{l\pi x'}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{m\pi y'}{b}\right)\sinh\left(\gamma_{lm} z_<\right)\sinh\left(\gamma_{lm}\left(c-z_>\right)\right)}{\gamma_{lm}\sinh\left(\gamma_{lm} c\right)}$$

where $\gamma_{lm} \equiv \sqrt{\left(\frac{l\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2}$

Other examples of Green's function constructions

Green's function for Poisson equation in spherical polar coordinates:

$$\nabla^2 \mathrm{G}(\mathbf{r},\mathbf{r}') = -4\pi\delta(\mathbf{r}-\mathbf{r}')$$

$$\mathrm{G}(\mathbf{r},\mathbf{r}') = 4\pi\sum_{lm}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}} Y_{lm}^*(\hat{\mathbf{r}}')Y_{lm}(\hat{\mathbf{r}})$$

Green's function for Helmholz equation in spherical polar coordinates:

$$\left(\nabla^2 + k^2\right)\mathrm{G}(\mathbf{r},\mathbf{r}') = -4\pi\delta(\mathbf{r}-\mathbf{r}')$$

$$\mathrm{G}(\mathbf{r},\mathbf{r}') = ik\sum_{lm} j_l(kr_<)h_l(kr_>)Y_{lm}^*(\hat{\mathbf{r}}')Y_{lm}(\hat{\mathbf{r}})$$

spherical Hankel function

spherical Bessel function