

**PHY 711 Classical Mechanics and
Mathematical Methods**
10-10:50 AM MWF Olin 103

Plan for Lecture 3:

1. Scattering theory – some more details
2. Rutherford scattering
3. Transformation between lab and center of mass reference frame.

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Scattering theory:

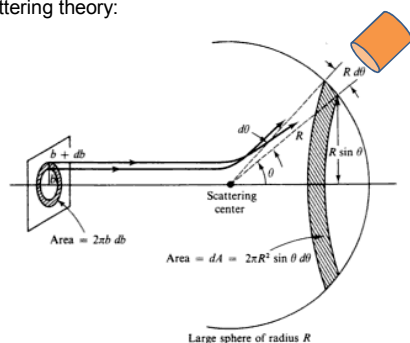


Figure 5.5 The scattering problem and relation of cross section to impact parameter.

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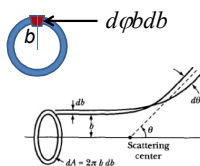
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Differential cross section

$$\left(\frac{d\sigma}{d\Omega} \right) = \frac{\text{Number of detected particles at } \theta \text{ per target particle}}{\text{Number of incident particles per unit area}}$$

= Area of incident beam that is scattered into detector at angle θ



$$\left(\frac{d\sigma}{d\Omega} \right) = \frac{d\phi b db}{d\phi \sin \theta d\theta} = \frac{b}{\sin \theta} \left| \frac{db}{d\theta} \right|$$

Figure from Marion & Thorton, Classical Dynamics

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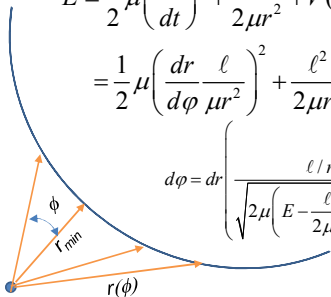
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In center of mass frame:

$$E = \frac{1}{2} \mu \left(\frac{dr}{dt} \right)^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

$$= \frac{1}{2} \mu \left(\frac{dr}{d\phi} \frac{\ell}{\mu r^2} \right)^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

$$d\phi = dr \left(\frac{\ell / r^2}{\sqrt{2\mu \left(E - \frac{\ell^2}{2\mu r^2} - V(r) \right)}} \right)$$


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Some details: conservation of angular momentum:

$$\ell = \mu r^2 \left(\frac{d\phi}{dt} \right)$$

Transformation of trajectory variables:

$$r(t) \Leftrightarrow r(\phi)$$

$$\frac{dr}{dt} = \frac{dr}{d\phi} \frac{d\phi}{dt} = \frac{dr}{d\phi} \frac{\ell}{\mu r^2}$$

Conservation of energy in the center of mass frame:

$$E = \frac{1}{2} \mu \left(\frac{dr}{dt} \right)^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

$$= \frac{1}{2} \mu \left(\frac{dr}{d\phi} \frac{\ell}{\mu r^2} \right)^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

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$$\Rightarrow E = \frac{1}{2} \mu \left(\frac{dr}{dt} \right)^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

$$= \frac{1}{2} \mu \left(\frac{dr}{d\phi} \frac{\ell}{\mu r^2} \right)^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

Solving for $r(\phi) \Leftrightarrow \phi(r)$

$$\left(\frac{dr}{d\phi} \right)^2 = \left(\frac{2\mu r^4}{\ell^2} \right) \left(E - \frac{\ell^2}{2\mu r^2} - V(r) \right)$$

$$d\phi = dr \left(\frac{\ell / r^2}{\sqrt{2\mu \left(E - \frac{\ell^2}{2\mu r^2} - V(r) \right)}} \right)$$

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$$d\phi = dr \left(\frac{\ell / r^2}{\sqrt{2\mu \left(E - \frac{\ell^2}{2\mu r^2} - V(r) \right)}} \right)$$

Further simplification at large separation:

$$\ell = \mu v_{\infty} b$$

$$E = \frac{1}{2} \mu v_{\infty}^2$$

$$\Rightarrow \ell = \sqrt{2\mu E} b$$

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When the dust clears:

$$d\phi = dr \left(\frac{\ell / r^2}{\sqrt{2\mu \left(E - \frac{\ell^2}{2\mu r^2} - V(r) \right)}} \right)$$

$$d\phi = dr \left(\frac{b / r^2}{\sqrt{1 - \frac{b^2}{r^2} - \frac{V(r)}{E}}} \right)$$

$$\Rightarrow \phi_{\max}(b, E) = \phi(r \rightarrow \infty) - \phi(r = r_{\min})$$

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$$\int_0^{\phi_{\max}} d\phi = \int_{r_{\min}}^{\infty} dr \left(\frac{b / r^2}{\sqrt{1 - \frac{b^2}{r^2} - \frac{V(r)}{E}}} \right)$$

where :

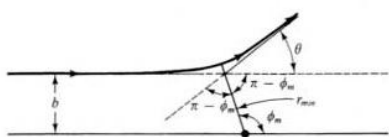
$$1 - \frac{b^2}{r_{\min}^2} - \frac{V(r_{\min})}{E} = 0$$

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Relationship between $\phi_{\max}$ and θ :



$$2(\pi - \phi_{\max}) + \theta = \pi$$

$$\Rightarrow \phi_{\max} = \frac{\pi}{2} + \frac{\theta}{2}$$

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$$\phi_{\max} = \frac{\pi}{2} + \frac{\theta}{2} = \int_{r_{\min}}^{\infty} dr \left(\frac{b/r^2}{\sqrt{1 - \frac{b^2}{r^2} - \frac{V(r)}{E}}} \right)$$

$$\theta = -\pi + 2b \int_{r_{\min}}^{\infty} dr \left(\frac{1/r^2}{\sqrt{1 - \frac{b^2}{r^2} - \frac{V(r)}{E}}} \right)$$

$$\theta = -\pi + 2b \int_0^{1/r_{\min}} du \left(\frac{1}{\sqrt{1 - b^2 u^2 - \frac{V(1/u)}{E}}} \right)$$

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Scattering angle equation:

$$\theta = -\pi + 2b \int_0^{1/r_{\min}} du \left(\frac{1}{\sqrt{1 - b^2 u^2 - \frac{V(1/u)}{E}}} \right)$$

where:

$$1 - \frac{b^2}{r_{\min}^2} - \frac{V(r_{\min})}{E} = 0$$

Rutherford scattering example:

$$\frac{V(r)}{E} \equiv \frac{\kappa}{r} \quad 1 - \frac{b^2}{r_{\min}^2} - \frac{\kappa}{r_{\min}} = 0$$

$$\frac{1}{r_{\min}} = \frac{1}{b} \left(-\frac{\kappa}{2b} + \sqrt{\left(\frac{\kappa}{2b} \right)^2 + 1} \right)$$

$$\theta = -\pi + 2b \int_0^{1/r_{\min}} du \left(\frac{1}{\sqrt{1 - b^2 u^2 - \kappa u}} \right) = 2 \sin^{-1} \left(\frac{1}{\sqrt{(2b/\kappa)^2 + 1}} \right)$$

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Rutherford scattering continued :

$$\theta = 2 \sin^{-1} \left(\frac{1}{\sqrt{(2b/\kappa)^2 + 1}} \right)$$

$$\frac{2b}{\kappa} = \left| \frac{\cos(\theta/2)}{\sin(\theta/2)} \right|$$

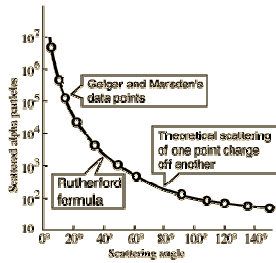
$$\left(\frac{d\sigma}{d\Omega} \right) = \frac{b}{\sin\theta} \left| \frac{db}{d\theta} \right| = \frac{\kappa^2}{16} \frac{1}{\sin^4(\theta/2)}$$

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$$\left(\frac{d\sigma}{d\Omega} \right) = \frac{b}{\sin\theta} \left| \frac{db}{d\theta} \right| = \frac{\kappa^2}{16} \frac{1}{\sin^4(\theta/2)}$$



What happens as $\theta \rightarrow 0$?

From webpage: <http://hyperphysics.phy-astr.gsu.edu/hbase/nuclear/rutsca2.html#c3>

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The results above were derived in the center of mass reference frame; relationship between normal laboratory reference and center of mass:

Laboratory reference frame:



Center of mass reference frame:



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Relationship between center of mass and laboratory frames of reference

Definition of center of mass $\mathbf{R}_{CM}$

$$m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2 = (m_1 + m_2) \mathbf{R}_{CM}$$

$$m_1 \dot{\mathbf{r}}_1 + m_2 \dot{\mathbf{r}}_2 = (m_1 + m_2) \dot{\mathbf{R}}_{CM}$$

$$m_1 \mathbf{u}_1 + m_2 \mathbf{u}_2 = (m_1 + m_2) \mathbf{V}_{CM} = m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2$$

In our case :

$$\mathbf{V}_{CM} = \frac{m_1}{m_1 + m_2} \mathbf{u}_1 = \frac{m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2}{m_1 + m_2}$$

$$\mathbf{u}_1 = \mathbf{U}_1 + \mathbf{V}_{CM} \quad \mathbf{u}_1 \quad \mathbf{V}_{CM} \quad \mathbf{v}_1 = \mathbf{V}_1 + \mathbf{V}_{CM} \quad \mathbf{V}_1 \quad \mathbf{V}_{CM} \quad \mathbf{v}_1$$

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Relationship between center of mass and laboratory frames of reference -- continued

Since m_2 is initially at rest :

$$\mathbf{V}_{CM} = \frac{m_1}{m_1 + m_2} \mathbf{u}_1 \quad \mathbf{u}_1 = \mathbf{U}_1 + \mathbf{V}_{CM} \Rightarrow \mathbf{U}_1 = \frac{m_2}{m_1 + m_2} \mathbf{u}_1 = \frac{m_2}{m_1} \mathbf{V}_{CM}$$

$$\mathbf{u}_2 = \mathbf{U}_2 + \mathbf{V}_{CM} \Rightarrow \mathbf{U}_2 = -\frac{m_1}{m_1 + m_2} \mathbf{u}_1 = -\mathbf{V}_{CM}$$

$$\mathbf{v}_1 = \mathbf{V}_1 + \mathbf{V}_{CM}$$

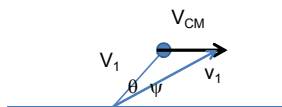
$$\mathbf{v}_2 = \mathbf{V}_2 + \mathbf{V}_{CM}$$

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Relationship between center of mass and laboratory frames of reference



$$\mathbf{v}_1 = \mathbf{V}_1 + \mathbf{V}_{CM}$$

$$v_1 \sin \psi = V_1 \sin \theta$$

$$v_1 \cos \psi = V_1 \cos \theta + V_{CM}$$

$$\tan \psi = \frac{\sin \theta}{\cos \theta + V_{CM} / V_1} = \frac{\sin \theta}{\cos \theta + m_1 / m_2}$$

For elastic scattering

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Digression – elastic scattering

$$\frac{1}{2} m_1 U_1^2 + \frac{1}{2} m_2 U_2^2 + \frac{1}{2} (m_1 + m_2) V_{CM}^2$$

$$= \frac{1}{2} m_1 V_1^2 + \frac{1}{2} m_2 V_2^2 + \frac{1}{2} (m_1 + m_2) V_{CM}^2$$

Also note:

$$m_1 \mathbf{U}_1 + m_2 \mathbf{U}_2 = 0 \quad m_1 \mathbf{V}_1 + m_2 \mathbf{V}_2 = 0$$

$$\mathbf{U}_1 = \frac{m_2}{m_1} \mathbf{V}_{CM} \quad \mathbf{U}_2 = -\mathbf{V}_{CM}$$

$$\Rightarrow |\mathbf{U}_1| = |\mathbf{V}_1| \quad \text{and} \quad |\mathbf{U}_2| = |\mathbf{V}_2| = |\mathbf{V}_{CM}|$$

$$\text{Also note that : } m_1 |\mathbf{U}_1| = m_2 |\mathbf{U}_2|$$

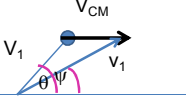
$$\text{So that : } V_{CM}/V_1 = V_{CM}/U_1 = m_1/m_2$$

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Relationship between center of mass and laboratory frames of reference – continued (elastic scattering)



$$\mathbf{v}_1 = \mathbf{V}_1 + \mathbf{V}_{CM}$$

$$v_1 \sin \psi = V_1 \sin \theta$$

$$v_1 \cos \psi = V_1 \cos \theta + V_{CM}$$

$$\tan \psi = \frac{\sin \theta}{\cos \theta + V_{CM}/V_1} = \frac{\sin \theta}{\cos \theta + m_1/m_2}$$

$$\text{Also : } \cos \psi = \frac{\cos \theta + m_1/m_2}{\sqrt{1 + 2(m_1/m_2) \cos \theta + (m_1/m_2)^2}}$$

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Differential cross sections in different reference frames

$$\left(\frac{d\sigma_{LAB}(\psi)}{d\Omega_{LAB}} \right) = \left(\frac{d\sigma_{CM}(\theta)}{d\Omega_{CM}} \right) \frac{d\Omega_{CM}}{d\Omega_{LAB}}$$

$$\frac{d\Omega_{CM}}{d\Omega_{LAB}} = \left| \frac{\sin \theta}{\sin \psi} \frac{d\theta}{d\psi} \right| = \left| \frac{d \cos \theta}{d \cos \psi} \right|$$

Using :

$$\cos \psi = \frac{\cos \theta + m_1/m_2}{\sqrt{1 + 2(m_1/m_2) \cos \theta + (m_1/m_2)^2}}$$

$$\left| \frac{d \cos \psi}{d \cos \theta} \right| = \frac{(m_1/m_2) \cos \theta + 1}{(1 + 2(m_1/m_2) \cos \theta + (m_1/m_2)^2)^{3/2}}$$

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Differential cross sections in different reference frames – continued:

$$\left(\frac{d\sigma_{LAB}(\psi)}{d\Omega_{LAB}} \right) = \left(\frac{d\sigma_{CM}(\theta)}{d\Omega_{CM}} \right) \left| \frac{d\cos\theta}{d\cos\psi} \right|$$

$$\left(\frac{d\sigma_{LAB}(\psi)}{d\Omega_{LAB}} \right) = \left(\frac{d\sigma_{CM}(\theta)}{d\Omega_{CM}} \right) \frac{(1 + 2m_1/m_2 \cos\theta + (m_1/m_2)^2)^{3/2}}{(m_1/m_2)\cos\theta + 1}$$

where : $\tan\psi = \frac{\sin\theta}{\cos\theta + m_1/m_2}$

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$$\left(\frac{d\sigma_{LAB}(\psi)}{d\Omega_{LAB}} \right) = \left(\frac{d\sigma_{CM}(\theta)}{d\Omega_{CM}} \right) \frac{(1 + 2m_1/m_2 \cos\theta + (m_1/m_2)^2)^{3/2}}{(m_1/m_2)\cos\theta + 1}$$

where : $\tan\psi = \frac{\sin\theta}{\cos\theta + m_1/m_2}$

Example: suppose $m_1 = m_2$

In this case : $\tan\psi = \frac{\sin\theta}{\cos\theta + 1} \Rightarrow \psi = \frac{\theta}{2}$

note that $0 \leq \psi \leq \frac{\pi}{2}$

$$\left(\frac{d\sigma_{LAB}(\psi)}{d\Omega_{LAB}} \right) = \left(\frac{d\sigma_{CM}(2\psi)}{d\Omega_{CM}} \right) \cdot 4\cos\psi$$

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Example of cross section analysis – CM versus lab frame

Rutherford scattering : $V(r) = \frac{\kappa E}{r} = \frac{Z_1 Z_2 e^2}{r}$

(Note that E is center of mass energy)

$$\left(\frac{d\sigma}{d\Omega} \right)_{CM} = \frac{\kappa^2}{16} \frac{1}{\sin^4(\theta/2)}$$

For $m_1 = m_2$

$$\left(\frac{d\sigma_{LAB}(\psi)}{d\Omega_{LAB}} \right) = \left(\frac{d\sigma_{CM}(2\psi)}{d\Omega_{CM}} \right) \cdot 4\cos\psi = \frac{\kappa^2}{4} \frac{\cos\psi}{\sin^4\psi}$$

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