PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

Plan for Lecture 4:

Chapter 2 – Physics described in an accelerated coordinate frame

1. Linear acceleration
2. Angular acceleration
3. Foucault pendulum

PHY 711 Classical Mechanics and Mathematical Methods

MWF 10 AM-10:50 PM	OPL 103	http://www.wfu.edu/~natalie/f15phy711/

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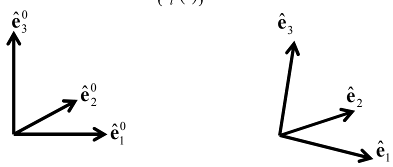
Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W Reading	Topic	Assignment
1	Wed, 8/26/2015	Chap. 1	Review of basic principles	#1
2	Fri, 8/28/2015	Chap. 1	Scattering theory	#2
3	Mon, 8/31/2015	Chap. 1	Scattering theory continued	#3
	Wed, 9/02/2015	Chap. 2	Accelerated coordinate systems	#4
5	Fri, 9/04/2015	Chap. 3	Calculus of variations	
6	Mon, 9/07/2015	Chap. 3	Calculus of variations	
7	Wed, 9/09/2015	Chap. 3	Hamilton's principle	

Physical laws as described in non-inertial coordinate systems

- Newton's laws are formulated in an inertial frame of reference $\left\{\hat{\mathbf{e}}_i^0\right\}$
- For some problems, it is convenient to transform the the equations into a non-inertial coordinate system $\left\{\hat{\mathbf{e}}_i(t)\right\}$



Comparison of analysis in "inertial frame" versus "non-inertial frame"

Denote by $\hat{e}_i^0$ a fixed coordinate system

Denote by $\hat{e}_i$ a moving coordinate system

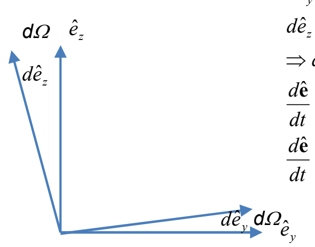
$$\mathbf{V} = \sum_{i=1}^{3} V_i^0 \hat{e}_i^0 = \sum_{i=1}^{3} V_i \hat{e}_i$$

$$\left(\frac{d\mathbf{V}}{dt}\right)_{inertial} = \sum_{i=1}^{3} \frac{dV_i^0}{dt}\hat{e}_i^0 = \sum_{i=1}^{3} \frac{dV_i}{dt}\hat{e}_i + \sum_{i=1}^{3} V_i \frac{d\hat{e}_i}{dt}$$

$$\text{Define : } \left(\frac{d\mathbf{V}}{dt}\right)_{body} \equiv \sum_{i=1}^{3} \frac{dV_i}{dt}\hat{e}_i$$

$$\Rightarrow \left(\frac{d\mathbf{V}}{dt}\right)_{inertial} = \left(\frac{d\mathbf{V}}{dt}\right)_{body} + \sum_{i=1}^{3} V_i \frac{d\hat{e}_i}{dt}$$

Properties of the frame motion (rotation only):



$$d\hat{e}_y = d\Omega \hat{e}_z$$

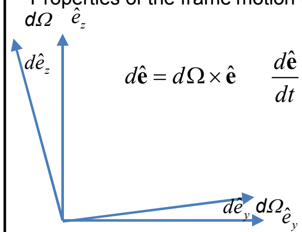
$$d\hat{e}_z = -d\Omega \hat{e}_y$$

$$\Rightarrow d\hat{\mathbf{e}} = d\mathbf{\Omega} \times \hat{\mathbf{e}}$$

$$\frac{d\hat{\mathbf{e}}}{dt} = \frac{d\mathbf{\Omega}}{dt} \times \hat{\mathbf{e}}$$

$$\frac{d\hat{\mathbf{e}}}{dt} = \boldsymbol{\omega} \times \hat{\mathbf{e}}$$

Properties of the frame motion (rotation only):



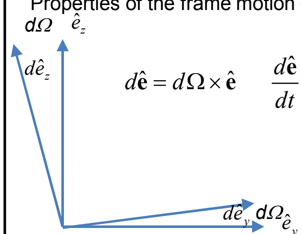
$$d\hat{\mathbf{e}} = d\Omega \times \hat{\mathbf{e}} \qquad \frac{d\hat{\mathbf{e}}}{dt} = \frac{d\Omega}{dt} \times \hat{\mathbf{e}} \qquad \frac{d\hat{\mathbf{e}}}{dt} = \boldsymbol{\omega} \times \hat{\mathbf{e}}$$

Rotation about x-axis:

$$\begin{pmatrix} e_y \\ e_z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e_y \\ e_z \end{pmatrix} \qquad \begin{pmatrix} e_y + de_y \\ e_z + de_z \end{pmatrix} = \begin{pmatrix} \cos(d\Omega) & -\sin(d\Omega) \\ \sin(d\Omega) & \cos(d\Omega) \end{pmatrix} \begin{pmatrix} e_y \\ e_z \end{pmatrix}$$

$$\begin{pmatrix} de_y \\ de_z \end{pmatrix} = \begin{pmatrix} \cos(d\Omega) - 1 & -\sin(d\Omega) \\ \sin(d\Omega) & \cos(d\Omega) - 1 \end{pmatrix} \begin{pmatrix} e_y \\ e_z \end{pmatrix} \approx \begin{pmatrix} 0 & -d\Omega \\ d\Omega & 0 \end{pmatrix} \begin{pmatrix} e_y \\ e_z \end{pmatrix}$$

Properties of the frame motion (rotation only):



$$d\hat{\mathbf{e}} = d\Omega \times \hat{\mathbf{e}} \qquad \frac{d\hat{\mathbf{e}}}{dt} = \frac{d\Omega}{dt} \times \hat{\mathbf{e}} \qquad \frac{d\hat{\mathbf{e}}}{dt} = \boldsymbol{\omega} \times \hat{\mathbf{e}}$$

Rotation about x-axis:

$$\begin{pmatrix} de_y \\ de_z \end{pmatrix} \approx \begin{pmatrix} 0 & -d\Omega \\ d\Omega & 0 \end{pmatrix} \begin{pmatrix} e_y \\ e_z \end{pmatrix} = d\Omega \hat{\mathbf{e}}_y - d\Omega \hat{\mathbf{e}}_z = d\Omega \hat{\mathbf{x}} \times \hat{\mathbf{e}}$$

Properties of the frame motion (rotation only) -- continued

$$\left(\frac{d\mathbf{V}}{dt}\right)_{inertial} = \left(\frac{d\mathbf{V}}{dt}\right)_{body} + \sum_{i=1}^{3} V_i \frac{d\hat{e}_i}{dt}$$

$$\left(\frac{d\mathbf{V}}{dt}\right)_{inertial} = \left(\frac{d\mathbf{V}}{dt}\right)_{body} + \boldsymbol{\omega} \times \mathbf{V}$$

Effects on acceleration (rotation only):

$$\left(\frac{d}{dt}\frac{d\mathbf{V}}{dt}\right)_{inertial} = \left(\left(\frac{d}{dt}\right)_{body} + \boldsymbol{\omega} \times\right)\left\{\left(\frac{d\mathbf{V}}{dt}\right)_{body} + \boldsymbol{\omega} \times \mathbf{V}\right\}$$

$$\left(\frac{d^2\mathbf{V}}{dt^2}\right)_{inertial} = \left(\frac{d^2\mathbf{V}}{dt^2}\right)_{body} + 2\boldsymbol{\omega} \times \left(\frac{d\mathbf{V}}{dt}\right)_{body} + \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{V} + \boldsymbol{\omega} \times \boldsymbol{\omega} \times \mathbf{V}$$

Application of Newton's laws in a coordinate system which has an angular velocity $\boldsymbol{\omega}$ and linear acceleration $\left(\frac{d^2\mathbf{a}}{dt^2}\right)_{inertial}$

Newton's laws; Let **r** denote the position of particle of mass *m*:

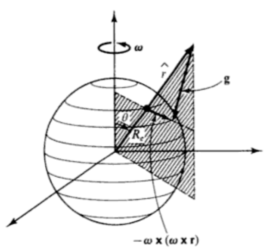
$$m\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{inertial} = \mathbf{F}_{ext}$$

$$m\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{body} = \mathbf{F}_{ext} - m\left(\frac{d^2\mathbf{a}}{dt^2}\right)_{inertial} - 2m\boldsymbol{\omega}\times\left(\frac{d\mathbf{r}}{dt}\right)_{body} - m\frac{d\boldsymbol{\omega}}{dt}\times\mathbf{r} - m\boldsymbol{\omega}\times\boldsymbol{\omega}\times\mathbf{r}$$

Coriolis force

Centrifugal force

Motion on the surface of the Earth:



$$\omega = \frac{2\pi}{\tau} \approx 7.3\times10^{-5}\,\text{rad/s}$$

$$\mathbf{F}_{ext} = -\frac{GM_e m}{r^2}\hat{\mathbf{r}} + \mathbf{F}'$$

Main contributions:

$$m\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{earth} = -\frac{GM_e m}{r^2}\hat{\mathbf{r}} + \mathbf{F}' - 2m\boldsymbol{\omega}\times\left(\frac{d\mathbf{r}}{dt}\right)_{earth} - m\frac{d\boldsymbol{\omega}}{dt}\times\mathbf{r} - m\boldsymbol{\omega}\times\boldsymbol{\omega}\times\mathbf{r}$$

Non-inertial effects on effective gravitational "constant"

$$m\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{earth} = -\frac{GM_e m}{r^2}\hat{\mathbf{r}} + \mathbf{F}' - 2m\boldsymbol{\omega}\times\left(\frac{d\mathbf{r}}{dt}\right)_{earth} - m\frac{d\boldsymbol{\omega}}{dt}\times\mathbf{r} - m\boldsymbol{\omega}\times\boldsymbol{\omega}\times\mathbf{r}$$

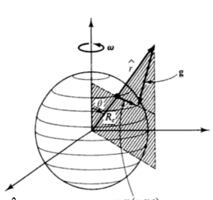
For $\left(\frac{d\mathbf{r}}{dt}\right)_{earth} = 0$ and $\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{earth} = 0,$

$$0 = -\frac{GM_e m}{r^2}\hat{\mathbf{r}} + \mathbf{F}' - m\boldsymbol{\omega}\times\boldsymbol{\omega}\times\mathbf{r}$$

$$\mathbf{F}' = -m\mathbf{g}$$

$$\Rightarrow \mathbf{g} = -\frac{GM_e}{r^2}\hat{\mathbf{r}} - \boldsymbol{\omega}\times\boldsymbol{\omega}\times\mathbf{r}\Big|_{r\approx R_e}$$

$$= \left(-\frac{GM_e}{R_e^2} + \omega^2 R_e \sin^2\theta\right)\hat{\mathbf{r}} + \sin\theta\cos\theta\,\omega^2 R_e\hat{\boldsymbol{\theta}}$$



0.03 m/s²

9.80 m/s²

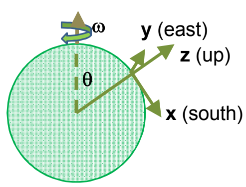
Foucault pendulum

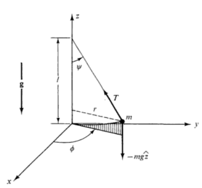
http://www.si.edu/Encyclopedia_SI/nmah/pendulum.htm

The Foucault pendulum was displayed for many years in the Smithsonian's National Museum of American History. It is named for the French physicist Jean Foucault who first used it in 1851 to demonstrate the rotation of the earth.

Equation of motion on Earth's surface

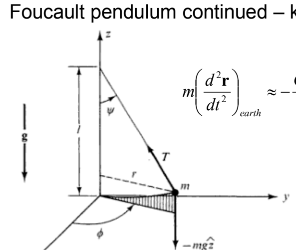
$$m\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{earth} = -\frac{GM_e m}{r^2}\hat{\mathbf{r}} + \mathbf{F}' - 2m\boldsymbol{\omega}\times\left(\frac{d\mathbf{r}}{dt}\right)_{earth} - m\frac{d\boldsymbol{\omega}}{dt}\times\mathbf{r} - m\boldsymbol{\omega}\times\boldsymbol{\omega}\times\mathbf{r}$$





$$\boldsymbol{\omega} \approx -\omega\sin\theta\hat{\mathbf{x}} + \omega\cos\theta\hat{\mathbf{z}}$$

Foucault pendulum continued – keeping leading terms:



$$m\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{earth} \approx -\frac{GM_e m}{R_e^2}\hat{\mathbf{r}} + \mathbf{F}' - 2m\boldsymbol{\omega}\times\left(\frac{d\mathbf{r}}{dt}\right)_{earth}$$

$$-\frac{GM_e m}{r^2}\hat{\mathbf{r}} \approx -mg\hat{\mathbf{z}}$$

$$\mathbf{F}' \approx -T\sin\psi\cos\phi\hat{\mathbf{x}} - T\sin\psi\sin\phi\hat{\mathbf{y}} + T\cos\psi\hat{\mathbf{z}}$$

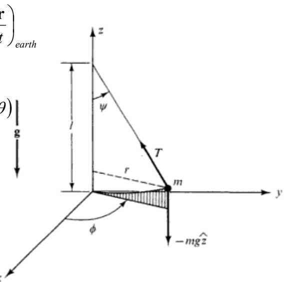
$$\boldsymbol{\omega} \approx -\omega\sin\theta\hat{\mathbf{x}} + \omega\cos\theta\hat{\mathbf{z}}$$

$$\boldsymbol{\omega}\times\left(\frac{d\mathbf{r}}{dt}\right)_{earth} \approx \omega\left(-\dot{y}\cos\theta\hat{\mathbf{x}} + \left(\dot{x}\cos\theta + \dot{z}\sin\theta\right)\hat{\mathbf{y}} - \dot{y}\sin\theta\hat{\mathbf{z}}\right)$$

Foucault pendulum continued – keeping leading terms:

$$m\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{earth} \approx -\frac{GM_e m}{R_e^{\ 2}}\hat{\mathbf{r}} + \mathbf{F}' - 2m\boldsymbol{\omega}\times\left(\frac{d\mathbf{r}}{dt}\right)_{earth}$$

$$m\ddot{x} \approx -T\sin\psi\cos\phi + 2m\omega\dot{y}\cos\theta$$
$$m\ddot{y} \approx -T\sin\psi\sin\phi - 2m\omega(\dot{x}\cos\theta + \dot{z}\sin\theta)$$
$$m\ddot{z} \approx T\cos\psi - mg + 2m\omega\dot{y}\sin s\theta$$



Further approximation :

$\psi << 1;\ \ \ddot{z} \approx 0;\ \ T \approx mg$

$m\ddot{x} \approx -mg\sin\psi\cos\phi + 2m\omega\dot{y}\cos\theta$

$m\ddot{y} \approx -mg\sin\psi\sin\phi - 2m\omega\dot{x}\cos\theta$

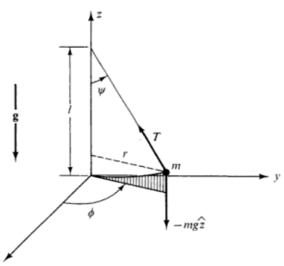
Also note that :

$x \approx \ell\sin\psi\cos\phi$

$y \approx \ell\sin\psi\sin\phi$

Foucault pendulum continued – coupled equations:

$$\ddot{x} \approx -\frac{g}{\ell}x + 2\omega\cos\theta\dot{y}$$
$$\ddot{y} \approx -\frac{g}{\ell}y - 2\omega\cos\theta\dot{x}$$



Try to find a solution of the form :

$x(t) = Xe^{-iqt} \quad y(t) = Ye^{-iqt}$

Denote $\omega_\perp \equiv \omega\cos\theta$

$$\begin{pmatrix} -q^2 + \frac{g}{\ell} & i2\omega_\perp q \\ -i2\omega_\perp q & -q^2 + \frac{g}{\ell}\end{pmatrix}\begin{pmatrix} X \\ Y\end{pmatrix} = 0$$

Non - trivial solutions :

$$q_\pm = \alpha \pm \beta \equiv \omega_\perp \pm \sqrt{\omega_\perp^{\ 2} + \frac{g}{\ell}}$$

Foucault pendulum continued – coupled equations:

Solution continued :

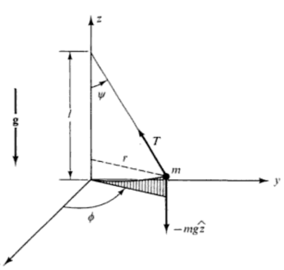
$x(t) = Xe^{-iqt} \quad y(t) = Ye^{-iqt}$

$$\begin{pmatrix} -q^2 + \frac{g}{\ell} & i2\omega_\perp q \\ -i2\omega_\perp q & -q^2 + \frac{g}{\ell}\end{pmatrix}\begin{pmatrix} X \\ Y\end{pmatrix} = 0$$

Non - trivial solutions :

$$q_\pm = \alpha \pm \beta \equiv \omega_\perp \pm \sqrt{\omega_\perp^{\ 2} + \frac{g}{\ell}}$$

Amplitude relationship : $X = iY$



General solution with complex amplitudes C and D :

$$x(t) = \mathrm{Re}\left\{iCe^{-i(\alpha+\beta)t} + iDe^{-i(\alpha-\beta)t}\right\}$$
$$y(t) = \mathrm{Re}\left\{Ce^{-i(\alpha+\beta)t} + De^{-i(\alpha-\beta)t}\right\}$$

General solution with complex amplitudes C and D:

$$x(t) = \mathrm{Re}\left\{iCe^{-i(\alpha+\beta)t} + iDe^{-i(\alpha-\beta)t}\right\}$$

$$y(t) = \mathrm{Re}\left\{Ce^{-i(\alpha+\beta)t} + De^{-i(\alpha-\beta)t}\right\}$$

$$q_{\pm} = \alpha \pm \beta \equiv \omega_{\perp} \pm \sqrt{\omega_{\perp}{}^{2} + \tfrac{g}{\ell}} \approx \omega_{\perp} \pm \sqrt{\tfrac{g}{\ell}}$$

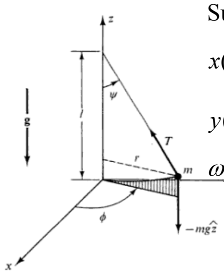
since $\quad \omega_{\perp} \approx 7\times10^{-5}\cos\theta \;\; rad/s << \sqrt{\tfrac{g}{\ell}}$

Suppose: $\quad x(0) = X_0 \qquad y(0) = 0$

$$x(t) = X_0 \cos\left(\sqrt{\tfrac{g}{\ell}}t\right)\cos(\omega_{\perp}t)$$

$$y(t) = -X_0 \cos\left(\sqrt{\tfrac{g}{\ell}}t\right)\sin(\omega_{\perp}t)$$

Summary of approximate solution for Foucault pendulum:



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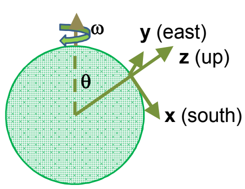
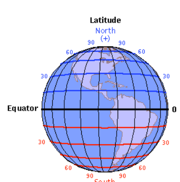
$$y(t) = -X_0 \cos\left(\sqrt{\tfrac{g}{\ell}}t\right)\sin(\omega_{\perp}t)$$

$$\omega_{\perp} \equiv \omega_0 \cos\theta$$





Microsoft Illustration

$$\omega_{\perp} \equiv \omega_0 \cos\theta$$

$$x(t) = X_0 \cos\left(\sqrt{\tfrac{g}{\ell}}t\right)\cos(\omega_{\perp}t)$$

$$y(t) = -X_0 \cos\left(\sqrt{\tfrac{g}{\ell}}t\right)\sin(\omega_{\perp}t)$$