PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

Plan for Lecture 6:

Continue reading Chapter 3

Further development of the "calculus of variation"

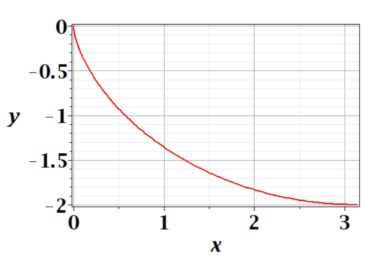
Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W Reading	Topic	Assignment
1	Wed, 8/26/2015	Chap. 1	Review of basic principles	#1
2	Fri, 8/28/2015	Chap. 1	Scattering theory	#2
3	Mon, 8/31/2015	Chap. 1	Scattering theory continued	#3
4	Wed, 9/02/2015	Chap. 2	Accelerated coordinate systems	#4
5	Fri, 9/04/2015	Chap. 3	Calculus of variations	#5
6	Mon, 9/07/2015	Chap. 3	Calculus of variations	#6
7	Wed, 9/09/2015	Chap. 3	Hamilton's principle	
8	Fri, 9/11/2015	Chap. 3 & 6	Hamilton's principle	
9	Mon, 9/14/2015	Chap. 3 & 6	Lagrangians with constraints	
10	Wed, 9/16/2015	Chap. 3 & 6	Lagrangians and constants of motion	
11	Fri, 9/18/2015	Chap. 3 & 6	Hamiltonian formalism	

Brachistochrone problem: (solved by Newton in 1696)

http://mathworld.wolfram.com/BrachistochroneProblem.html



A particle of weight *mg* travels frictionlessly down a path of shape *y(x)*. What is the shape of the path *y(x)* that minimizes the travel time from $y(0)=0$ to $y(\pi)=-2$?

$$T = \int_{x_i y_i}^{x_f y_f} \frac{ds}{v} = \int_{x_i}^{x_f} \frac{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\sqrt{-2gy}} dx \quad \text{because} \quad \tfrac{1}{2}mv^2 = -mgy$$

$$f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) = \sqrt{\frac{1+\left(\frac{dy}{dx}\right)^2}{-y}}$$

Alternative relationships for extremization:

$$\left(\frac{\partial f}{\partial y}\right) - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial (dy/dx)}\right)\right] = 0$$

$$\frac{d}{dx}\left(f - \frac{\partial f}{\partial (dy/dx)} \frac{dy}{dx}\right) = \left(\frac{\partial f}{\partial x}\right)$$

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$$f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) = \sqrt{\frac{1+\left(\frac{dy}{dx}\right)^2}{-y}}$$

$$\frac{d}{dx}\left(f - \frac{\partial f}{\partial (dy/dx)} \frac{dy}{dx}\right) = \left(\frac{\partial f}{\partial x}\right)$$

$$\Rightarrow \frac{d}{dx}\left(\frac{1}{\sqrt{-y\left(1+\left(\frac{dy}{dx}\right)^2\right)}}\right) = 0 \qquad -y\left(1+\left(\frac{dy}{dx}\right)^2\right) = K \equiv 2a$$

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$$-y\left(1+\left(\frac{dy}{dx}\right)^2\right) = K \equiv 2a \qquad \text{Let} \quad y = -2a\sin^2\tfrac{\theta}{2} = a(\cos\theta - 1)$$

$$\frac{dy}{dx} = -\sqrt{\frac{2a}{-y} - 1}$$

$$-\frac{dy}{\sqrt{\frac{2a}{-y} - 1}} = dx$$

$$-\frac{dy}{\sqrt{\frac{2a}{-y} - 1}} = \frac{2a\sin\frac{\theta}{2}\cos\frac{\theta}{2}\,d\theta}{\sqrt{\frac{2a}{2a\sin^2\frac{\theta}{2}} - 1}} = dx$$

$$x = \int_0^\theta a(1-\cos\theta')d\theta' = a(\theta - \sin\theta)$$

Parametric equations for Brachistochrone:

$$x = a(\theta - \sin\theta)$$
$$y = a(\cos\theta - 1)$$

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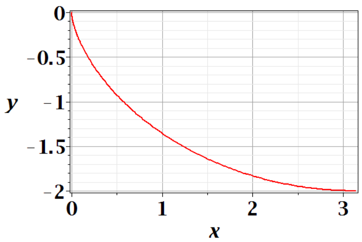
Brachistochrone problem -- summary



$$-y\left(1+\left(\frac{dy}{dx}\right)^2\right)=K\equiv 2a$$

$$\frac{dy}{dx}=-\sqrt{\frac{2a}{-y}-1}$$

Check :

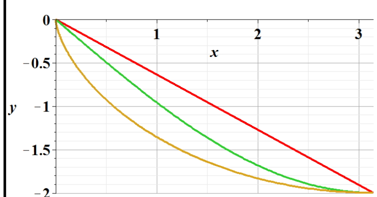
$$\frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{-a\sin\theta}{a(1-\cos\theta)}$$
$$=-\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}$$

$$-\sqrt{\frac{2a}{-y}-1}=-\sqrt{\frac{y+2a}{-y}}$$
$$=-\sqrt{\frac{(\cos\theta+1)}{-\cos\theta+1}}$$
$$=-\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}$$

Parametric equations; $0\le\theta\le\pi$

$$x=a(\theta-\sin\theta)$$
$$y=a(\cos\theta-1)$$

Brachistochrone problem -- summary



Check : $y_1(x)=-\frac{2}{\pi}x$

$$T_1\sqrt{2g}=\left(\sqrt{1+\left(\tfrac{2}{\pi}\right)^2}\right)\sqrt{2a\pi}$$
$$=1.185\sqrt{2a\pi}$$

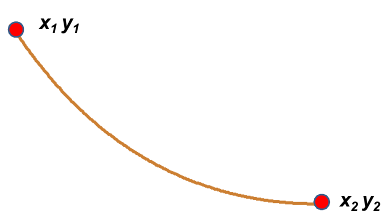
Check : For optimal $y(x)$:

$$T\sqrt{2g}=\int_0^{\pi a}\sqrt{\frac{1+\left(\frac{dy}{dx}\right)^2}{-y}}\,dx=\sqrt{2a\pi}$$

Check : $y_2(x)=-2a\sin\left(\frac{x}{2a}\right)$

$$T_2\sqrt{2g}=\left(\sqrt{1+\left(\tfrac{2}{\pi}\right)^2}\right)\sqrt{2a\pi}$$
$$=2.378\sqrt{2a\pi}$$

Shape of a rope of length L and mass density ρ hanging between two points



Potential energy of hanging rope:

$$E = \rho g \int_{x_1}^{x_2} y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Length of rope:

$$L = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Define a composite function to minimize:

$$W \equiv E + \lambda L$$

Lagrange multiplier

$$W = \int_{x_1}^{x_2} (\rho g y + \lambda) \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$f\left(\left\{y, \frac{dy}{dx}\right\}\right) = (\rho g y + \lambda) \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$\frac{d}{dx}\left(f - \frac{\partial f}{\partial (dy/dx)} \frac{dy}{dx}\right) = \left(\frac{\partial f}{\partial x}\right)$$

$$\Rightarrow (\rho g y + \lambda) \left(\sqrt{1 + \left(\frac{dy}{dx}\right)^2} - \frac{\left(\frac{dy}{dx}\right)^2}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}} \right) = K$$

$$(\rho g y + \lambda) \left(\sqrt{1 + \left(\frac{dy}{dx}\right)^2} - \frac{\left(\frac{dy}{dx}\right)^2}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}} \right) = K$$

$$(\rho g y + \lambda) \left(\frac{1}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}} \right) = K$$

$$y(x) = -\frac{1}{\rho g}\left(\lambda + K \cosh\left(\frac{x - a}{K / \rho g}\right)\right)$$

$$y(x) = -\frac{1}{\rho g}\left(\lambda + K\cosh\left(\frac{x-a}{K/\rho g}\right)\right)$$

Integration constants : K, a, λ

Constraints : $y(x_1) = y_1$

$$y(x_2) = y_2$$

$$\int_{x_1}^{x_2}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = L$$

Summary of results

For the class of problems where we need to perform an extremization on an integral form :

$$I = \int_{x_i}^{x_f} f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right)dx \qquad \delta I = 0$$

A necessary condition is the Euler - Lagrange equations :

$$\left(\frac{\partial f}{\partial y}\right) - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)\right] = 0$$

$$\frac{d}{dx}\left(f - \frac{\partial f}{\partial(dy/dx)}\frac{dy}{dx}\right) = \left(\frac{\partial f}{\partial x}\right)$$

Application to particle dynamics

$x \to t$ (time)

$y \to q$ (generalized coordinate)

$f \to L$ (Lagrangian)

$I \to A$ (action)

Denote : $\dot{q} \equiv \frac{dq}{dt}$

$$A = \int_{t_1}^{t_2} L(\{q, \dot{q}\}; t)\,dt$$