

**PHY 711 Classical Mechanics and  
Mathematical Methods**  
**10-10:50 AM MWF Olin 103**

**Plan for Lecture 7:**

**Continue reading Chapter 3**

**1. Lagrange's equations**

**2. D'Alembert's principle**

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**PHY 711 Classical Mechanics and Mathematical Methods**

**MWF 10 AM-10:50 PM | OPL 103 | <http://www.wfu.edu/~natalie/f15phy711/>**

**Instructor: [Natalie Holzwarth](mailto:natalie@wfu.edu) Phone: 758-5510 Office: 300 OPL e-mail: [natalie@wfu.edu](mailto:natalie@wfu.edu)**

**Course schedule**

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W Reading	Topic	Assignment
1	Wed, 8/26/2015	Chap. 1	Review of basic principles	<a href="#">#1</a>
2	Fri, 8/28/2015	Chap. 1	Scattering theory	<a href="#">#2</a>
3	Mon, 8/31/2015	Chap. 1	Scattering theory continued	<a href="#">#3</a>
4	Wed, 9/02/2015	Chap. 2	Accelerated coordinate systems	<a href="#">#4</a>
5	Fri, 9/04/2015	Chap. 3	Calculus of variations	<a href="#">#5</a>
6	Mon, 9/07/2015	Chap. 3	Calculus of variations	<a href="#">#6</a>
7	Wed, 9/09/2015	Chap. 3	Hamilton's principle	<a href="#">#7</a>
8	Fri, 9/11/2015	Chap. 3 & 6	Hamilton's principle	
9	Mon, 9/14/2015	Chap. 3 & 6	Lagrangians with constraints	
10	Wed, 9/16/2015	Chap. 3 & 6	Lagrangians and constants of motion	

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Department of Physics

**News**



[Research Labs Tour Part I](#)



Congratulations to Dr.  
Greg Smith, recent Ph.D.  
Recipient



Congratulations to Dr. Jie  
Liu, recent Ph.D. Recipient

**Events**

Wed. Sept. 9, 2015  
[Laboratory tours for  
students -- Part II](#)  
**Meet in Olin Lobby at 4 PM**  
Refreshments at 3:30 PM  
Olin Lobby

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**WFU Physics Colloquium****TITLE:** "Laboratory tours for students -- Part II"**TIME:** Wednesday Sept. 9, 2015 at 4:00 PM**PLACE:** Olin Lounge

Refreshments will be served **for everyone** at 3:30 PM in the lounge. All interested persons are cordially invited to attend.

**PROGRAM**

This is the second of three opportunities that students will have to learn about the research which is being done in the Physics Department. Students will meet in the Olin lobby and divide into three groups. Each group will tour the labs of Professors Guthold, Kim-Shapiro, and Macosko.

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**Summary of results from the calculus of variation**

For the class of problems where we need to perform an extremization on an integral form :

$$I = \int_{x_i}^{x_f} f\left(y(x), \frac{dy}{dx}, x\right) dx$$

A necessary condition is the Euler - Lagrange equations :

$$\left(\frac{\partial f}{\partial y}\right) - \frac{d}{dx} \left[ \left(\frac{\partial f}{\partial (dy/dx)}\right) \right] = 0$$

$$\frac{d}{dx} \left( f - \frac{\partial f}{\partial (dy/dx)} \frac{dy}{dx} \right) = \left(\frac{\partial f}{\partial x}\right)$$

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**Application to particle dynamics**

Simple example: vertical trajectory of particle of mass  $m$  subject to constant downward acceleration  $a=-g$ .

$$m \frac{d^2 y}{dt^2} = -mg$$

$$y(t) = y_i + v_i t - \frac{1}{2} g t^2$$

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<http://www.hamilton2005.ie/>

**Sir William Rowan Hamilton**

Wednesday, September 11th, 2015



**Tribute to Sir William Hamilton**

Hello and welcome! This page is dedicated to the life and work of Sir William Rowan Hamilton.

William Rowan Hamilton was Ireland's greatest scientist. He was an mathematician, physicist, and astronomer and made important works in optics, dynamics, and algebra.

His contribution in dynamics plays a important role in the later developed quantum mechanics. His name was perpetuated in one of the fundamental concepts in quantum mechanics, called "Hamiltonian".

The Discovery of Quaternions is probably is his most familar invention today.

2005 was the Hamilton Year, celebrating his 200th birthday. The year was dedicated to celebrate Irish Science. 2005 was called the Einstein year also, reminding of three great papers of the year 1905. So UNESCO designated 2005 to the World Year of Physics

Thanks for visiting this site! Please enjoy your stay while browsing through the pages.

**Sitemap**

- Home
- Biography
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- Hamilton Key Dates
- Hamilton Links
- Graphics
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<http://rjlipton.wordpress.com>

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Now consider the Lagrangian defined to be :

$$L\left(\left\{y(t), \frac{dy}{dt}\right\}, t\right) \equiv T - U$$

Kinetic energy
Potential energy

In our example :

$$L\left(\left\{y(t), \frac{dy}{dt}\right\}, t\right) \equiv T - U = \frac{1}{2} m \left(\frac{dy}{dt}\right)^2 - mgy$$

Hamilton's principle states :

$$S \equiv \int_{t_i}^{t_f} \left( \frac{1}{2} m \left(\frac{dy}{dt}\right)^2 - mgy \right) dt \text{ is minimized for physical } y(t):$$

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Condition for minimizing the action :

$$S \equiv \int_{t_i}^{t_f} \left( \frac{1}{2} m \left( \frac{dy}{dt} \right)^2 - mgy \right) dt$$

Euler - Lagrange relations :

$$\frac{\partial L}{\partial y} - \frac{d}{dt} \frac{\partial L}{\partial \dot{y}} = 0$$

$$\Rightarrow -mg - \frac{d}{dt} m\dot{y} = 0$$

$$\Rightarrow \frac{d}{dt} \frac{dy}{dt} = -g$$

$$y(t) = y_i + v_i t - \frac{1}{2} g t^2$$

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Check:

$$S \equiv \int_{t_i}^{t_f} \left( \frac{1}{2} m \left( \frac{dy}{dt} \right)^2 - mgy \right) dt$$

Assume  $t_i = 0$ ,  $y_i = h \equiv \frac{1}{2} g T^2$ ;  $t_f = T$ ,  $y_f = 0$

Trial trajectories:  $y_1(t) = \frac{1}{2} g T^2 (1 - t / T) = h - \frac{1}{2} g T t$

$$y_2(t) = \frac{1}{2} g T^2 (1 - t^2 / T^2) = h - \frac{1}{2} g t^2$$

$$y_3(t) = \frac{1}{2} g T^2 (1 - t^3 / T^3) = h - \frac{1}{2} g t^3 / T$$

Maple says:

$$S_1 = -0.125 m g^2 T^3$$

$$S_2 = -0.167 m g^2 T^3$$

$$S_3 = -0.150 m g^2 T^3$$

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Jean d'Alembert 1717-1783

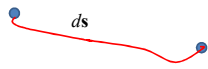


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D'Alembert's principle:



Generalized coordinates :  
 $q_\sigma(\{x_i\})$

Newton's laws :

$$\mathbf{F} \cdot \mathbf{ma} = 0 \quad \Rightarrow \quad (\mathbf{F} - \mathbf{ma}) \cdot d\mathbf{s} = 0$$

$$\mathbf{F} \cdot d\mathbf{s} = \sum_\sigma \sum_i F_i \frac{\partial x_i}{\partial q_\sigma} \delta q_\sigma$$

For a conservative force :  $F_i = -\frac{\partial U}{\partial x_i}$

$$\mathbf{F} \cdot d\mathbf{s} = -\sum_\sigma \sum_i \frac{\partial U}{\partial x_i} \frac{\partial x_i}{\partial q_\sigma} \delta q_\sigma = -\sum_\sigma \frac{\partial U}{\partial q_\sigma} \delta q_\sigma$$

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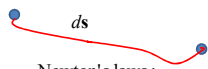
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Generalized coordinates :  
 $q_\sigma(\{x_i\})$

Newton's laws :

$$\mathbf{F} \cdot \mathbf{ma} = 0 \quad \Rightarrow \quad (\mathbf{F} - \mathbf{ma}) \cdot d\mathbf{s} = 0$$

$$\mathbf{ma} \cdot d\mathbf{s} = \sum_\sigma \sum_i m \ddot{x}_i \frac{\partial x_i}{\partial q_\sigma} \delta q_\sigma$$

$$= \sum_\sigma \sum_i \left( \frac{d}{dt} \left( m \dot{x}_i \frac{\partial x_i}{\partial q_\sigma} \right) - m \dot{x}_i \frac{d}{dt} \frac{\partial x_i}{\partial q_\sigma} \right) \delta q_\sigma$$

Claim :  $\frac{\partial x_i}{\partial q_\sigma} = \frac{\partial \dot{x}_i}{\partial \dot{q}_\sigma}$  and  $\frac{d}{dt} \frac{\partial x_i}{\partial q_\sigma} = \frac{\partial}{\partial q_\sigma} \frac{dx_i}{dt} \equiv \frac{\partial \dot{x}_i}{\partial q_\sigma}$

$$\mathbf{ma} \cdot d\mathbf{s} = \sum_\sigma \sum_i \left( \frac{d}{dt} \left( \frac{\partial}{\partial \dot{q}_\sigma} \left( \frac{1}{2} m \dot{x}_i^2 \right) \right) - \frac{\partial}{\partial q_\sigma} \left( \frac{1}{2} m \dot{x}_i^2 \right) \right) \delta q_\sigma$$

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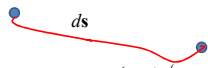
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Generalized coordinates :  
 $q_\sigma(\{x_i\})$

$$\mathbf{ma} \cdot d\mathbf{s} = \sum_\sigma \sum_i \left( \frac{d}{dt} \left( \frac{\partial}{\partial \dot{q}_\sigma} \left( \frac{1}{2} m \dot{x}_i^2 \right) \right) - \frac{\partial}{\partial q_\sigma} \left( \frac{1}{2} m \dot{x}_i^2 \right) \right) \delta q_\sigma$$

Define -- kinetic energy :  $T \equiv \sum_i \frac{1}{2} m \dot{x}_i^2$

$$\mathbf{ma} \cdot d\mathbf{s} = \sum_\sigma \left( \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_\sigma} - \frac{\partial T}{\partial q_\sigma} \right) \delta q_\sigma$$

Recall :

$$\mathbf{F} \cdot d\mathbf{s} = \sum_\sigma \sum_i \frac{\partial U}{\partial x_i} \frac{\partial x_i}{\partial q_\sigma} \delta q_\sigma = \sum_\sigma \frac{\partial U}{\partial q_\sigma} \delta q_\sigma$$

$$(\mathbf{F} - \mathbf{ma}) \cdot d\mathbf{s} = \sum_\sigma \frac{\partial U}{\partial q_\sigma} \delta q_\sigma - \sum_\sigma \left( \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_\sigma} - \frac{\partial T}{\partial q_\sigma} \right) \delta q_\sigma = 0$$

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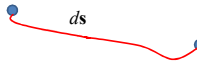
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Generalized coordinates :  
 $q_\sigma(\{x_i\})$

$$(\mathbf{F}-m\mathbf{a}) \cdot d\mathbf{s} = -\sum_\sigma \frac{\partial U}{\partial q_\sigma} \delta q_\sigma - \sum_\sigma \left( \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_\sigma} - \frac{\partial T}{\partial q_\sigma} \right) \delta q_\sigma = 0$$

$$= -\sum_\sigma \left( \frac{d}{dt} \frac{\partial (T-U)}{\partial \dot{q}_\sigma} - \frac{\partial (T-U)}{\partial q_\sigma} \right) \delta q_\sigma = 0$$

$$L(q_\sigma, \dot{q}_\sigma; t) = T - U$$

Note : This is only true if  
 $\frac{\partial U}{\partial \dot{q}_\sigma} = 0$

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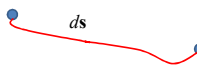
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Generalized coordinates :  
 $q_\sigma(\{x_i\})$

Define -- Lagrangian :  $L = T - U$   
 $L = L(\{q_\sigma\}, \{\dot{q}_\sigma\}, t)$

$$(\mathbf{F}-m\mathbf{a}) \cdot d\mathbf{s} = -\sum_\sigma \left( \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} \right) \delta q_\sigma = 0$$

$$\Rightarrow \text{Minimization integral : } S = \int_{t_i}^{t_f} L(\{q_\sigma\}, \{\dot{q}_\sigma\}, t) dt$$

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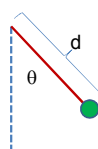
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Euler – Lagrange equations :  $L = L(\{q_\sigma\}, \{\dot{q}_\sigma\}, t) = T - U$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

Example:



$$L = L(\theta, \dot{\theta}) = \frac{1}{2} m d^2 \dot{\theta}^2 - mg(d - d \cos \theta)$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0 \Rightarrow \frac{d}{dt} m d^2 \dot{\theta} - mg d \sin \theta = 0$$

$$\frac{d^2 \theta}{dt^2} = \frac{g}{d} \sin \theta$$

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Another example:  $L = L(\{q_\sigma\}, \{\dot{q}_\sigma\}, t) = T - U$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

$$L = L(\alpha, \beta, \gamma, \dot{\alpha}, \dot{\beta}, \dot{\gamma}) = \frac{1}{2} I_1 (\dot{\alpha}^2 \sin^2 \beta + \dot{\beta}^2) + \frac{1}{2} I_3 (\dot{\alpha} \cos \beta + \dot{\gamma})^2 - Mgd \cos \beta$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\alpha}} = \frac{d}{dt} (I_1 \dot{\alpha} \sin^2 \beta + I_3 (\dot{\alpha} \cos \beta + \dot{\gamma}) \cos \beta) = 0$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\beta}} = \frac{d}{dt} (I_1 \dot{\beta}) = \text{mess}$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\gamma}} = \frac{d}{dt} (I_3 (\dot{\alpha} \cos \beta + \dot{\gamma})) = 0$$

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