

**PHY 711 Classical Mechanics and
Mathematical Methods**
10-10:50 AM MWF Olin 103

Plan for Lecture 8:

Continue reading Chapter 3 & 6

- 1. D'Alembert's principle**
- 2. Hamilton's principle**
- 3. Lagrange's equations in
presence of magnetic fields**

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PHY 711 Classical Mechanics and Mathematical Methods

MWF 10 AM-10:50 PM OPL 103 (<http://www.wfu.edu/~natalie/f15phy711/>)

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Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

Date	F&W Reading	Topic	Assignment
1 Wed, 8/26/2015	Chap. 1	Review of basic principles	#1
2 Fri, 8/28/2015	Chap. 1	Scattering theory	#2
3 Mon, 8/31/2015	Chap. 1	Scattering theory continued	#3
4 Wed, 9/02/2015	Chap. 2	Accelerated coordinate systems	#4
5 Fri, 9/04/2015	Chap. 3	Calculus of variations	#5
6 Mon, 9/07/2015	Chap. 3	Calculus of variations	#6
7 Wed, 9/09/2015	Chap. 3	Hamilton's principle	#7
8 Fri, 9/11/2015	Chap. 3 & 6	Hamilton's principle	#8
9 Mon, 9/14/2015	Chap. 3 & 6	Lagrangians with constraints	
10 Wed, 9/16/2015	Chap. 3 & 6	Lagrangians and constants of motion	

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Hamilton's principle:

Given the Lagrangian function: $L = L(\{q_\sigma\}, \{\dot{q}_\sigma\}, t) \equiv T - U$,

The physical trajectories of the generalized coordinates $\{q_\sigma(t)\}$

Are those which minimize the action: $S = \int L(\{q_\sigma\}, \{\dot{q}_\sigma\}, t) dt$

Euler-Lagrange equations:

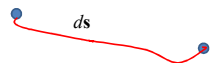
$$\sum_\sigma \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} \right) \delta q_\sigma = 0 \quad \Rightarrow \text{for each } \sigma: \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} \right) = 0$$

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D'Alembert's principle -- notion of virtual work:



Generalized coordinates:
 $q_\sigma(\{x_i\})$

Newton's laws:

$$\mathbf{F} - m\mathbf{a} = 0 \quad \Rightarrow (\mathbf{F} - m\mathbf{a}) \cdot d\mathbf{s} = 0$$

$$m\mathbf{a} \cdot d\mathbf{s} = \sum_\sigma \sum_i m\dot{x}_i \frac{\partial x_i}{\partial q_\sigma} \delta q_\sigma$$

$$= \sum_\sigma \sum_i \left(\frac{d}{dt} \left(m\dot{x}_i \frac{\partial x_i}{\partial q_\sigma} \right) - m\dot{x}_i \frac{d}{dt} \frac{\partial x_i}{\partial q_\sigma} \right) \delta q_\sigma$$

Claim: $\frac{\partial x_i}{\partial q_\sigma} = \frac{\partial \dot{x}_i}{\partial \dot{q}_\sigma}$ and $\frac{d}{dt} \frac{\partial x_i}{\partial q_\sigma} = \frac{\partial}{\partial q_\sigma} \frac{dx_i}{dt} \equiv \frac{\partial \dot{x}_i}{\partial q_\sigma}$

$$m\mathbf{a} \cdot d\mathbf{s} = \sum_\sigma \sum_i \left(\frac{d}{dt} \left(\frac{\partial (\frac{1}{2} m \dot{x}_i^2)}{\partial \dot{q}_\sigma} \right) - \frac{\partial (\frac{1}{2} m \dot{x}_i^2)}{\partial q_\sigma} \right) \delta q_\sigma$$

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Some details --

$$m\mathbf{a} \cdot d\mathbf{s} = \sum_\sigma \sum_i m\dot{x}_i \frac{\partial x_i}{\partial q_\sigma} \delta q_\sigma \quad \text{where } x_i(\{q_\sigma(t)\}, t)$$

$$= \sum_\sigma \sum_i \left(\frac{d}{dt} \left(m\dot{x}_i \frac{\partial x_i}{\partial q_\sigma} \right) - m\dot{x}_i \frac{d}{dt} \frac{\partial x_i}{\partial q_\sigma} \right) \delta q_\sigma$$

Claim: $\frac{\partial x_i}{\partial q_\sigma} = \frac{\partial \dot{x}_i}{\partial \dot{q}_\sigma}$ and $\frac{d}{dt} \frac{\partial x_i}{\partial q_\sigma} = \frac{\partial}{\partial q_\sigma} \frac{dx_i}{dt} \equiv \frac{\partial \dot{x}_i}{\partial q_\sigma}$

Details: $\frac{dx_i}{dt} = \sum_\sigma \frac{\partial x_i}{\partial q_\sigma} \frac{dq_\sigma}{dt} + \frac{\partial x_i}{\partial t} = \sum_\sigma \frac{\partial x_i}{\partial q_\sigma} \dot{q}_\sigma + \frac{\partial x_i}{\partial t}$

$$\Rightarrow \frac{\partial \dot{x}_i}{\partial \dot{q}_\sigma} = \frac{\partial x_i}{\partial q_\sigma}$$

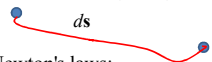
$$\Rightarrow \frac{\partial \dot{x}_i}{\partial q_\sigma} = \sum_\sigma \frac{\partial^2 x_i}{\partial q_\sigma \partial q_\sigma} \dot{q}_\sigma + \frac{\partial^2 x_i}{\partial q_\sigma \partial t} = \frac{d}{dt} \left(\frac{\partial x_i}{\partial q_\sigma} \right)$$

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D'Alembert's principle -- notion of virtual work:



Generalized coordinates:
 $q_\sigma(\{x_i\})$

Newton's laws:

$$\mathbf{F} - m\mathbf{a} = 0 \quad \Rightarrow (\mathbf{F} - m\mathbf{a}) \cdot d\mathbf{s} = 0$$

$$m\mathbf{a} \cdot d\mathbf{s} = \sum_\sigma \sum_i \left(\frac{d}{dt} \left(\frac{\partial (\frac{1}{2} m \dot{x}_i^2)}{\partial \dot{q}_\sigma} \right) - \frac{\partial (\frac{1}{2} m \dot{x}_i^2)}{\partial q_\sigma} \right) \delta q_\sigma$$

$$= \sum_\sigma \left(\frac{d}{dt} \left(\frac{\partial (T)}{\partial \dot{q}_\sigma} \right) - \frac{\partial (T)}{\partial q_\sigma} \right) \delta q_\sigma \quad \text{where } T \equiv \sum_i \left(\frac{1}{2} m \dot{x}_i^2 \right)$$

$$\mathbf{F} \cdot d\mathbf{s} = \sum_\sigma \sum_i F_i \frac{\partial x_i}{\partial q_\sigma} \delta q_\sigma = - \sum_\sigma \sum_i \frac{\partial U}{\partial x_i} \frac{\partial x_i}{\partial q_\sigma} \delta q_\sigma = - \sum_\sigma \frac{\partial U}{\partial q_\sigma} \delta q_\sigma$$

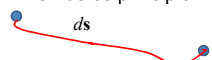
$$(\mathbf{F} - m\mathbf{a}) \cdot d\mathbf{s} = - \sum_\sigma \left(\frac{d}{dt} \left(\frac{\partial (T - U)}{\partial \dot{q}_\sigma} \right) - \frac{\partial (T - U)}{\partial q_\sigma} \right) \delta q_\sigma = 0$$

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D'Alembert's principle -- notion of virtual work:



Generalized coordinates :

$$q_\sigma(\{x_i\})$$

$$(\mathbf{F} - m\mathbf{a}) \cdot d\mathbf{s} = - \sum_\sigma \left(\frac{d}{dt} \left(\frac{\partial(T-U)}{\partial \dot{q}_\sigma} \right) - \frac{\partial(T-U)}{\partial q_\sigma} \right) \delta q_\sigma = 0$$

Provided that $\frac{\partial U}{\partial \dot{q}_\sigma} = 0$

Consistent with Hamilton's principle with

$$L = T - U = L(\{q_\sigma\}, \{\dot{q}_\sigma\}, t)$$

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Example -- simple harmonic oscillator

$$T = \frac{1}{2} m \dot{x}^2$$

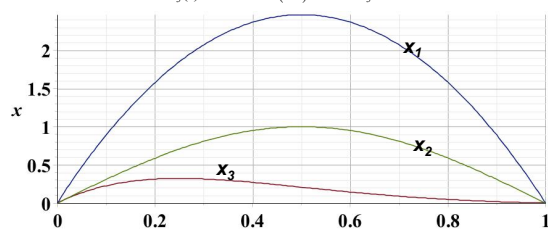
$$U = \frac{1}{2} m \omega^2 x^2$$

Assume $x(0) = 0$ and $x(\frac{\pi}{\omega}) = 0$ $S = \int_0^{\pi/\omega} (T - U) dt$

Trial functions $x_1(t) = A \sin(\omega t)$ $S_1 = 0$

$x_2(t) = A \omega t \cdot (\pi - \omega t)$ $S_2 = 0.067 A^2 m \omega^2$

$x_3(t) = A e^{-\omega t} \sin(\omega t)$ $S_3 = 0.062 A^2 m \omega^2$



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Note: in "proof" of Hamilton's principle:

$$\left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} \right) = 0 \quad \text{for} \quad L = L(\{q_\sigma\}, \{\dot{q}_\sigma\}, t) \equiv T - U$$

It was necessary to assume that :

$$\frac{d}{dt} \frac{\partial U}{\partial \dot{q}_\sigma} \text{ does not contribute to the result.}$$

$\Rightarrow$ How can we represent velocity - dependent forces?

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Lorentz forces:

For particle of charge q in an electric field $\mathbf{E}(\mathbf{r}, t)$ and magnetic field $\mathbf{B}(\mathbf{r}, t)$:

Lorentz force: $\mathbf{F} = q(\mathbf{E} + \frac{1}{c} \mathbf{v} \times \mathbf{B})$

x -component: $F_x = q(E_x + \frac{1}{c}(\mathbf{v} \times \mathbf{B})_x)$

In this case, it is convenient to use cartesian coordinates

$$L = L(x, y, z, \dot{x}, \dot{y}, \dot{z}, t) \equiv T - U$$

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

$$x\text{-component: } \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} \right) = 0$$

$$\text{Apparently: } F_x = -\frac{\partial U}{\partial x} + \frac{d}{dt} \frac{\partial U}{\partial \dot{x}}$$

$$\text{Answer: } U = q\Phi(\mathbf{r}, t) - \frac{q}{c} \dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}, t)$$

$$\text{where } \mathbf{E}(\mathbf{r}, t) = -\nabla\Phi(\mathbf{r}, t) - \frac{1}{c} \frac{\partial \mathbf{A}(\mathbf{r}, t)}{\partial t} \quad \mathbf{B}(\mathbf{r}, t) = \nabla \times \mathbf{A}(\mathbf{r}, t)$$

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Lorentz forces, continued:

x -component of Lorentz force: $F_x = q(E_x + \frac{1}{c}(\mathbf{v} \times \mathbf{B})_x)$

$$\text{Suppose: } U = q\Phi(\mathbf{r}, t) - \frac{q}{c} \dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}, t)$$

$$\text{Consider: } F_x = -\frac{\partial U}{\partial x} + \frac{d}{dt} \frac{\partial U}{\partial \dot{x}}$$

$$-\frac{\partial U}{\partial x} = -q \frac{\partial \Phi(\mathbf{r}, t)}{\partial x} + \frac{q}{c} \left(\dot{x} \frac{\partial A_x(\mathbf{r}, t)}{\partial x} + \dot{y} \frac{\partial A_y(\mathbf{r}, t)}{\partial x} + \dot{z} \frac{\partial A_z(\mathbf{r}, t)}{\partial x} \right)$$

$$\frac{\partial U}{\partial \dot{x}} = -\frac{q}{c} A_x(\mathbf{r}, t)$$

$$\frac{d}{dt} \frac{\partial U}{\partial \dot{x}} = -\frac{q}{c} \frac{dA_x(\mathbf{r}, t)}{dt} = -\frac{q}{c} \left(\frac{\partial A_x(\mathbf{r}, t)}{\partial x} \dot{x} + \frac{\partial A_x(\mathbf{r}, t)}{\partial y} \dot{y} + \frac{\partial A_x(\mathbf{r}, t)}{\partial z} \dot{z} + \frac{\partial A_x(\mathbf{r}, t)}{\partial t} \right)$$

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Lorentz forces, continued:

$$-\frac{\partial U}{\partial x} = -q \frac{\partial \Phi(\mathbf{r}, t)}{\partial x} + \frac{q}{c} \left(\dot{x} \frac{\partial A_x(\mathbf{r}, t)}{\partial x} + \dot{y} \frac{\partial A_y(\mathbf{r}, t)}{\partial x} + \dot{z} \frac{\partial A_z(\mathbf{r}, t)}{\partial x} \right)$$

$$\frac{d}{dt} \frac{\partial U}{\partial \dot{x}} = -\frac{q}{c} \left(\frac{\partial A_x(\mathbf{r}, t)}{\partial x} \dot{x} + \frac{\partial A_x(\mathbf{r}, t)}{\partial y} \dot{y} + \frac{\partial A_x(\mathbf{r}, t)}{\partial z} \dot{z} + \frac{\partial A_x(\mathbf{r}, t)}{\partial t} \right)$$

$$\begin{aligned} F_x &= -\frac{\partial U}{\partial x} + \frac{d}{dt} \frac{\partial U}{\partial \dot{x}} \\ &= -q \frac{\partial \Phi(\mathbf{r}, t)}{\partial x} + \frac{q}{c} \dot{y} \left(\frac{\partial A_y(\mathbf{r}, t)}{\partial x} - \frac{\partial A_x(\mathbf{r}, t)}{\partial y} \right) + \frac{q}{c} \dot{z} \left(\frac{\partial A_z(\mathbf{r}, t)}{\partial x} - \frac{\partial A_x(\mathbf{r}, t)}{\partial z} \right) - \frac{q}{c} \frac{\partial A_x(\mathbf{r}, t)}{\partial t} \\ &= -q \frac{\partial \Phi(\mathbf{r}, t)}{\partial x} - \frac{q}{c} \frac{\partial A_x(\mathbf{r}, t)}{\partial t} + \frac{q}{c} \dot{y} \left(\frac{\partial A_y(\mathbf{r}, t)}{\partial x} - \frac{\partial A_x(\mathbf{r}, t)}{\partial y} \right) + \frac{q}{c} \dot{z} \left(\frac{\partial A_z(\mathbf{r}, t)}{\partial x} - \frac{\partial A_x(\mathbf{r}, t)}{\partial z} \right) \\ &= qE_x(\mathbf{r}, t) + \frac{q}{c} (\dot{y}B_z(\mathbf{r}, t) - \dot{z}B_y(\mathbf{r}, t)) = qE_x(\mathbf{r}, t) + \frac{q}{c} (\mathbf{v} \times \mathbf{B}(\mathbf{r}, t))_x \end{aligned}$$

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Lorentz forces, continued:

Summary of results (using cartesian coordinates)

$$L = L(x, y, z, \dot{x}, \dot{y}, \dot{z}, t) \equiv T - U$$

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \quad U = q\Phi(\mathbf{r}, t) - \frac{q}{c} \dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}, t)$$

$$\text{where } \mathbf{E}(\mathbf{r}, t) = -\nabla\Phi(\mathbf{r}, t) - \frac{1}{c} \frac{\partial \mathbf{A}(\mathbf{r}, t)}{\partial t} \quad \mathbf{B}(\mathbf{r}, t) = \nabla \times \mathbf{A}(\mathbf{r}, t)$$

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - q\Phi(\mathbf{r}, t) + \frac{q}{c} \dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}, t)$$

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Example Lorentz force

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - q\Phi(\mathbf{r}, t) + \frac{q}{c} \dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}, t)$$

$$\text{Suppose } \mathbf{E}(\mathbf{r}, t) \equiv 0, \quad \mathbf{B}(\mathbf{r}, t) \equiv B_0 \hat{\mathbf{z}}$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{1}{2} B_0 (-y\hat{\mathbf{x}} + x\hat{\mathbf{y}})$$

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{q}{2c} B_0 (-\dot{x}y + \dot{y}x)$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \quad \Rightarrow \quad \frac{d}{dt} \left(m\dot{x} - \frac{q}{2c} B_0 y \right) - \frac{q}{2c} B_0 \dot{y} = 0$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = 0 \quad \Rightarrow \quad \frac{d}{dt} \left(m\dot{y} + \frac{q}{2c} B_0 x \right) + \frac{q}{2c} B_0 \dot{x} = 0$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{z}} - \frac{\partial L}{\partial z} = 0 \quad \Rightarrow \quad \frac{d}{dt} m\dot{z} = 0$$

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Example Lorentz force -- continued

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{q}{2c} B_0 (-\dot{x}y + \dot{y}x)$$

$$\frac{d}{dt} \left(m\dot{x} - \frac{q}{2c} B_0 y \right) - \frac{q}{2c} B_0 \dot{y} = 0 \quad \Rightarrow \quad m\ddot{x} - \frac{q}{c} B_0 \dot{y} = 0$$

$$\frac{d}{dt} \left(m\dot{y} + \frac{q}{2c} B_0 x \right) + \frac{q}{2c} B_0 \dot{x} = 0 \quad \Rightarrow \quad m\ddot{y} + \frac{q}{c} B_0 \dot{x} = 0$$

$$\frac{d}{dt} m\dot{z} = 0 \quad \Rightarrow \quad m\ddot{z} = 0$$

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Example Lorentz force -- continued

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{q}{2c} B_0 (-\dot{x}y + \dot{y}x)$$

$$m\ddot{x} = +\frac{q}{c} B_0 \dot{y}$$

$$m\ddot{y} = -\frac{q}{c} B_0 \dot{x}$$

$$m\ddot{z} = 0$$

Note that same equations are obtained
from direct application of Newton's laws :

$$m\ddot{\mathbf{r}} = \frac{q}{c} \dot{\mathbf{r}} \times B_0 \hat{\mathbf{z}}$$

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Example Lorentz force -- continued

Consider formulation with different Gauge : $\mathbf{A}(\mathbf{r}) = -B_0 y \hat{\mathbf{x}}$

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \frac{q}{c} B_0 \dot{x}y$$

$$\frac{d}{dt} \left(m\dot{x} - \frac{q}{c} B_0 y \right) = 0 \quad \Rightarrow \quad m\ddot{x} - \frac{q}{c} B_0 \dot{y} = 0$$

$$\frac{d}{dt} (m\dot{y}) + \frac{q}{c} B_0 \dot{x} = 0 \quad \Rightarrow \quad m\ddot{y} + \frac{q}{c} B_0 \dot{x} = 0$$

$$\frac{d}{dt} m\dot{z} = 0 \quad \Rightarrow \quad m\ddot{z} = 0$$

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Example Lorentz force -- continued

Evaluation of equations :

$$m\ddot{x} - \frac{q}{c} B_0 \dot{y} = 0 \quad \dot{x}(t) = V_0 \sin\left(\frac{qB_0}{mc} t + \phi\right)$$

$$m\ddot{y} + \frac{q}{c} B_0 \dot{x} = 0 \quad \dot{y}(t) = V_0 \cos\left(\frac{qB_0}{mc} t + \phi\right)$$

$$m\ddot{z} = 0 \quad \dot{z}(t) = V_{0z}$$

$$x(t) = x_0 - \frac{mc}{qB_0} V_0 \cos\left(\frac{qB_0}{mc} t + \phi\right)$$

$$y(t) = y_0 + \frac{mc}{qB_0} V_0 \sin\left(\frac{qB_0}{mc} t + \phi\right)$$

$$z(t) = z_0 + V_{0z} t$$

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