

PHY 711 Classical Mechanics and Mathematical Methods

10-10:50 AM MWF Olin 103

Plan for Lecture 9:

Continue reading Chapter 3 & 6

1. Summary & review

2. Lagrange's equations with constraints

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PHY 711 Classical Mechanics and Mathematical Methods

MWF 10 AM-10:50 PM OPL 103 <http://www.wfu.edu/~natalie/f15phy711/>

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Course schedule

(Preliminary schedule – subject to frequent adjustment.)

Date	F&W Reading Topic	Assignment
1 Wed, 8/26/2015	Chap. 1 Review of basic principles	#1
2 Fri, 8/28/2015	Chap. 1 Scattering theory	#2
3 Mon, 8/31/2015	Chap. 1 Scattering theory continued	#3
4 Wed, 9/02/2015	Chap. 2 Accelerated coordinate systems	#4
5 Fri, 9/04/2015	Chap. 3 Calculus of variations	#5
6 Mon, 9/07/2015	Chap. 3 Calculus of variations	#6
7 Wed, 9/09/2015	Chap. 3 Hamilton's principle	#7
8 Fri, 9/11/2015	Chap. 3 & 6 Hamilton's principle	#8
9 Mon, 9/14/2015	Chap. 3 & 6 Lagrangians with constraints	#9
10 Wed, 9/16/2015	Chap. 3 & 6 Lagrangians and constants of motion	
11 Fri, 9/18/2015	Chap. 3 & 6 Hamiltonian formalism	
12 Mon, 9/21/2015	Chap. 3 & 6 Hamiltonian formalism	

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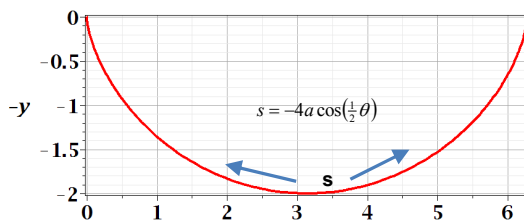
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Comment on problem Lagrangian formulation of Brachistochrone motion:

$$x(\theta) = a(\theta - \sin \theta)$$

$$y(\theta) = a(1 - \cos \theta)$$



Lagrangian for mass traveling along s :

$$L(s(t), \dot{s}(t)) = \frac{1}{2} m \dot{s}^2 - mgy = \frac{1}{2} m \dot{s}^2 - mg2a \left(1 - \left(\frac{s}{4a} \right)^2 \right)$$

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Lagrangian for mass traveling along s :

$$L(s(t), \dot{s}(t)) = \frac{1}{2} m \dot{s}^2 - mgy = \frac{1}{2} m \dot{s}^2 - mg2a \left(1 - \left(\frac{s}{4a} \right)^2 \right)$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{s}} - \frac{\partial L}{\partial s} = 0$$

$$\Rightarrow m\ddot{s} = -\frac{mg}{4a}s$$

$$\Rightarrow \ddot{s} = -\frac{g}{4a}s$$

$$s(t) = s_0 + A \sin \left(\sqrt{\frac{g}{4a}} t \right)$$

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Comments on generalized coordinates:

$$L = L(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t)$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

Here we have assumed that the generalized coordinates q_σ are independent. Now consider the possibility that the coordinates are related through constraint equations of the form:

Lagrangian: $L = L(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t)$

Constraints: $f_j = f_j(\{q_\sigma(t)\}, t) = 0$

Lagrange
multipliers

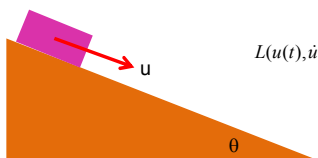
Modified Euler-Lagrange equations: $\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} + \sum_j \lambda_j \frac{\partial f_j}{\partial q_\sigma} = 0$

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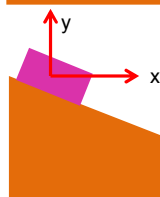
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Simple example:



$$L(u(t), \dot{u}(t)) = \frac{1}{2} m \dot{u}^2 + mgu \sin \theta$$



$$L(x, y, \dot{x}, \dot{y}) = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) - mgy$$

$$f(x, y) = \sin \theta x + \cos \theta y = 0$$

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Case 1:

$$L(u(t), \dot{u}(t)) = \frac{1}{2} m \dot{u}^2 + m g u \sin \theta$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{u}} - \frac{\partial L}{\partial u} = 0 = m \ddot{u} - m g \sin \theta = 0$$

Case 2: $\Rightarrow \ddot{u} = g \sin \theta$

$$L(x, y, \dot{x}, \dot{y}) = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) - m g y$$

$$f(x, y) = \sin \theta x + \cos \theta y = 0$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} + \lambda \frac{\partial f}{\partial x} = 0 = m \ddot{x} + \lambda \sin \theta$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} + \lambda \frac{\partial f}{\partial y} = 0 = m \ddot{y} + m g + \lambda \cos \theta$$

$$\sin \theta \ddot{x} + \cos \theta \ddot{y} = 0$$

$$\Rightarrow \lambda = m g \cos \theta$$

$$(-\cos \theta \ddot{x} + \sin \theta \ddot{y}) = -g \sin \theta$$

Force of constraint;
normal to incline

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Rational for Lagrange multipliers

Recall Hamilton's principle :

$$S = \int_{t_i}^{t_f} L(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t) dt$$

$$\delta S = 0 = \int_{t_i}^{t_f} \left(\sum_\sigma \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} \right) \delta q_\sigma \right) dt$$

With constraints: $f_j = f_j(\{q_\sigma(t)\}, t) = 0$ Variations δq_σ are no longer independent.

$$\delta f_j = 0 = \sum_\sigma \frac{\partial f_j}{\partial q_\sigma} \delta q_\sigma \quad \text{at each } t$$

 $\Rightarrow$ Add 0 to Euler - Lagrange equations in the form :

$$\sum_j \lambda_j \sum_\sigma \frac{\partial f_j}{\partial q_\sigma} \delta q_\sigma$$

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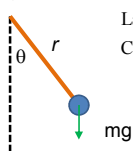
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Euler-Lagrange equations with constraints:

Lagrangian: $L = L(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t)$ Constraints: $f_j = f_j(\{q_\sigma(t)\}, t) = 0$ Modified Euler - Lagrange equations: $\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} + \sum_j \lambda_j \frac{\partial f_j}{\partial q_\sigma} = 0$

Example:

Lagrangian: $L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) + m g r \cos \theta$ Constraints: $f = r - \ell = 0$

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Example continued:

Lagrangian : $L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) + mgr \cos \theta$

Constraints : $f = r - \ell = 0$

$$\frac{d}{dt} m \dot{r} - m r \dot{\theta}^2 - mg \cos \theta + \lambda = 0$$

$$\frac{d}{dt} m r^2 \dot{\theta} + mgr \sin \theta = 0$$

$$\dot{r} = 0 = \ddot{r} \quad r = \ell$$

$$\Rightarrow \ddot{\theta} = -\frac{g}{\ell} \sin \theta$$

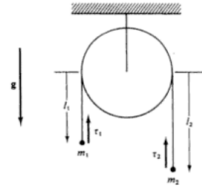
$$\Rightarrow \lambda = m \ell \dot{\theta}^2 + mg \cos \theta$$

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Another example:



Lagrangian : $L = \frac{1}{2} m_1 \dot{\ell}_1^2 + \frac{1}{2} m_2 \dot{\ell}_2^2 + m_1 g \ell_1 + m_2 g \ell_2$

Constraints : $f = \ell_1 + \ell_2 - \ell = 0$

$$\frac{d}{dt} m_1 \dot{\ell}_1 - m_1 g + \lambda = 0$$

$$\frac{d}{dt} m_2 \dot{\ell}_2 - m_2 g + \lambda = 0$$

$$\dot{\ell}_1 + \dot{\ell}_2 = 0 = \ddot{\ell}_1 + \ddot{\ell}_2$$

$$\Rightarrow \lambda = \frac{2m_1 m_2}{m_1 + m_2} g$$

$$\ddot{\ell}_1 = -\ddot{\ell}_2 = \frac{m_1 - m_2}{m_1 + m_2} g$$

Figure 19.1 Atwood's machine.

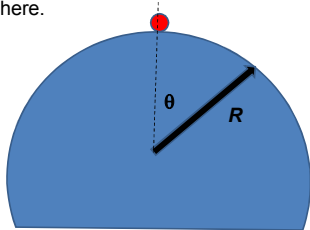
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Another example:

A particle of mass m starts at rest on top of a smooth fixed hemisphere of radius R . Find the angle at which the particle leaves the hemisphere.



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Example continued

Constraint Equation : $f(r, \theta) = r - R$

Lagrangian : $L(r, \theta, \dot{r}, \dot{\theta}) = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) - mgr \cos \theta$

Euler - Lagrangian equations :

$$\frac{\partial L}{\partial r} - \frac{d}{dt} \frac{\partial L}{\partial \dot{r}} + \lambda \frac{\partial f}{\partial r} = 0 \quad mr\dot{\theta}^2 - mg \cos \theta - m\ddot{r} + \lambda = 0$$

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} + \lambda \frac{\partial f}{\partial \theta} = 0 \quad mgr \sin \theta - mr^2 \ddot{\theta} - 2mr\dot{\theta} = 0$$

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Example continued

$$mr\dot{\theta}^2 - mg \cos \theta - m\ddot{r} + \lambda = 0$$

$$mgr \sin \theta - mr^2 \ddot{\theta} - 2mr\dot{\theta} = 0$$

Using constraint:

$$mR\dot{\theta}^2 - mg \cos \theta + \lambda = 0$$

$$mgR \sin \theta - mR^2 \ddot{\theta} = 0$$

$$\ddot{\theta} = \frac{g}{R} \sin \theta \quad \Rightarrow \dot{\theta}^2 = -\frac{2g}{R} (\cos \theta - 1)$$

$$\Rightarrow \lambda = mg(3 \cos \theta - 2)$$

When $\lambda \leq 0$, constraint is no longer in place

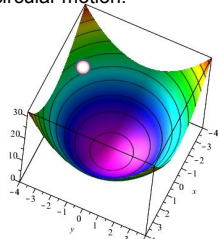
$$\text{at } \cos \theta < \frac{2}{3} \quad \text{or } \theta > 48.2 \text{ deg}$$

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Consider a particle of mass m moving frictionlessly on a parabola $z=c(x^2+y^2)$ under the influence of gravity. Find the equations of motion, particularly showing stable circular motion.



$$L(x, y, \dot{x}, \dot{y}) = \frac{m}{2} (\dot{x}^2 + \dot{y}^2 + 4c^2 (\dot{x}\dot{y})^2) - mgc(x^2 + y^2)$$

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$$L(x, y, \dot{x}, \dot{y}) = \frac{m}{2} (\dot{x}^2 + \dot{y}^2 + 4c^2 (x\dot{x} + y\dot{y})^2) - mgc(x^2 + y^2)$$

Transform to polar coordinates;

$$x = r \cos \phi \quad y = r \sin \phi$$

$$L(r, \phi, \dot{r}, \dot{\phi}) = \frac{m}{2} (\dot{r}^2 + r^2 \dot{\phi}^2 + 4c^2 r^2 \dot{r}^2) - mgcr^2$$

Euler-Lagrange equations

$$\frac{\partial L}{\partial \phi} - \frac{d}{dt} \frac{\partial L}{\partial \dot{\phi}} = 0 \Rightarrow 0 - \frac{d}{dt} mr^2 \dot{\phi} = 0$$

$$\frac{\partial L}{\partial r} - \frac{d}{dt} \frac{\partial L}{\partial \dot{r}} = 0 \Rightarrow \text{Let } mr^2 \dot{\phi} \equiv \ell_z \text{ (constant)}$$

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$$L(r, \phi, \dot{r}, \dot{\phi}) = \frac{m}{2} (\dot{r}^2 + r^2 \dot{\phi}^2 + 4c^2 r^2 \dot{r}^2) - mgcr^2$$

$$\frac{\partial L}{\partial r} - \frac{d}{dt} \frac{\partial L}{\partial \dot{r}} = 0$$

$$mr\dot{\phi}^2 + 4mr^2 c^2 \dot{r} - 2mgcr - \frac{d}{dt} (mr^2 (1 + 4c^2 r^2)) = 0$$

$$\frac{\ell_z^2}{mr^3} - 2mgcr - \frac{d}{dt} (mr^2 (1 + 2c^2 r^2)) = 0$$

Stable solution when these terms add to 0:

$$\ell_z = mr_0^2 \sqrt{2gc} = m z_0 \sqrt{\frac{2g}{c}}$$

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Analysis of stable (circular) motion

$$\frac{\ell_z^2}{mr^3} - 2mgcr - \frac{d}{dt} (mr^2 (1 + 2c^2 r^2)) = 0$$

$$\ell_z = mr_0^2 \sqrt{2gc} = m z_0 \sqrt{\frac{2g}{c}}$$

Let $r = r_0 + \delta r$ keeping terms to linear order:

$$-8mgc\delta r - m\delta\ddot{r}(1 + 2c^2 r_0^2) = 0$$

$$\delta\ddot{r} = -\frac{8gc}{1 + 2c^2 r_0^2} \delta r$$

$$\Rightarrow \delta r = A \cos \left(\sqrt{\frac{8gc}{1 + 2c^2 r_0^2}} t + \alpha \right)$$

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