PHY 752 Solid State Physics
11-11:50 AM MWF Olin 103

Plan for Lecture 6:

Reading: Chapter 2 in GGGPP;

Continued brief introduction to group theory

1. **Group multiplication tables**
2. **Representations of groups**
3. **The "great" orthogonality theorem**

PHY 752 Solid State Physics

MWF 11 AM-11:50 PM	OPL 103	http://www.wfu.edu/~natalie/f15phy752/

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Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W Reading	Topic	Assignment
1	Wed, 8/26/2015	Chap. 1.1-1.2	Electrons in a periodic one-dimensional potential	#1
2	Fri, 8/28/2015	Chap. 1.3	Electrons in a periodic one-dimensional potential	#2
3	Mon, 8/31/2015	Chap. 1.4	Tight binding models	#3
4	Wed, 9/02/2015	Chap. 1.6, 2.1	Crystal structures	#4
5	Fri, 9/04/2015	Chap. 2	Group theory	#5
6	Mon, 9/07/2015	Chap. 2	Group theory	#6
7	Wed, 9/09/2015	Chap. 2	Group theory	
8	Fri, 9/11/2015			
9	Mon, 9/14/2015			
10	Wed, 9/16/2015			

Short digression on abstract group theory
What is group theory ?

A group is a collection of "elements" – $A, B, C, \ldots$ and a "multiplication" process. The abstract multiplication $(\cdot)$ pairs two group elements, and associates the "result" with a third element. (For example $(A \cdot B = C)$.) The elements and the multiplication process must have the following properties.

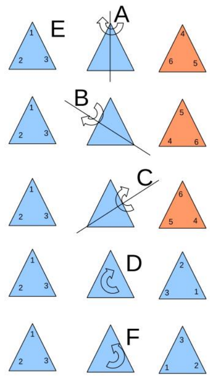
1. The collection of elements is closed under multiplication. That is, if elements A and B are in the group and $A \cdot B = C$, element C must be in the group.
2. One of the members of the group is a "unit element" (E). That is, for any element A of the group, $A \cdot E = E \cdot A = A$.
3. For each element A of the group, there is another element A^{-1} which is its "inverse". That is $A \cdot A^{-1} = A^{-1} \cdot A = E$.
4. The multiplication process is "associative". That is for sequential mulplication of group elements A, B, and C, $(A \cdot B) \cdot C = A \cdot (B \cdot C)$.

Example of a 6-member group *E,A,B,C,D,F,G*

Group multiplication table

Group of order 6

	E	A	B	C	D	F
E	E	A	B	C	D	F
A	A	E	D	F	B	C
B	B	F	E	D	C	A
C	C	D	F	E	A	B
D	D	C	A	B	F	E
F	F	B	C	A	E	D



	E	A	B	C	D	F
E	E	A	B	C	D	F
A	A	E	D	F	B	C
B	B	F	E	D	C	A
C	C	D	F	E	A	B
D	D	C	A	B	F	E
F	F	B	C	A	E	D

Check on group properties:

1. Closed; multiplication table uniquely generates group members.
2. Unit element included.
3. Each element has inverse.
4. Multiplication process is associative.

Definitions

Subgroup: members of larger group which have the property of a group

Class: members of a group which are generated by the construction

$$\mathcal{C} = X_i^{-1} Y X_i \quad \text{where } X_i \text{ and } Y \text{ are group elements}$$

Group theory – some comments

- The elements of the group may be abstract; in general, we will use them to describe symmetry properties of our system

Representations of a group

A representation of a group is a set of matrices (one for each group element) -- $\Gamma(A), \Gamma(B)$... that satisfies the multiplication table of the group. The dimension of the matrices is called the dimension of the representation.

Example:

	E	A	B	C	D	F
E	E	A	B	C	D	F
A	A	E	D	F	B	C
B	B	F	E	D	C	A
C	C	D	F	E	A	B
D	D	C	A	B	F	E
F	F	B	C	A	E	D

Note that the one-dimensional "identical representation"

$\Gamma^1(A) = \Gamma^1(B) = \Gamma^1(C) = \Gamma^1(D) = \Gamma^1(E) = \Gamma^1(F) = 1$ is always possible

Another one-dimensional representation is

$$\Gamma^2(A) = \Gamma^2(B) = \Gamma^2(C) = -1$$

$$\Gamma^2(E) = \Gamma^2(D) = \Gamma^2(F) = 1$$

Example:

	E	A	B	C	D	F
E	E	A	B	C	D	F
A	A	E	D	F	B	C
B	B	F	E	D	C	A
C	C	D	F	E	A	B
D	D	C	A	B	F	E
F	F	B	C	A	E	D

A two-dimensional representation is

$$\Gamma^3(E) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \Gamma^3(A) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\Gamma^3(B) = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \quad \Gamma^3(C) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

$$\Gamma^3(D) = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \quad \Gamma^3(F) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

What about 3 or 4 dimensional representations for this group?

$$\Gamma(E) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \Gamma(A) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \Gamma(B) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

$$\Gamma(C) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \quad \Gamma(D) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \quad \Gamma(F) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

The only "irreducible" representations for this group are 2 one-dimensional and 1 two-dimensional

Comment about representation matrices

A representation is not fundamentally altered by a similarity transformation

$\Gamma'(A) = S^{-1}\Gamma(A)S$

Check:

$\Gamma'(AB) = S^{-1}\Gamma(AB)S = S^{-1}\Gamma(A)\Gamma(B)S$

$= S^{-1}\Gamma(A)SS^{-1}\Gamma(B)S$

$= \Gamma'(A)\Gamma'(B)$

- Typically, unitary matrices are chosen for representations
- Typically representations are reduced to block diagonal form and the irreducible blocks are considered in the representation theory

The great orthogonality theorem

Notation: $h \equiv$ order of the group

$R \equiv$ element of the group

$\Gamma^i(R)_{\alpha\beta} \equiv$ ith representation of R

$_{\alpha\beta}$ denote matrix indices

$l_i \equiv$ dimension of the representation

$$\sum_R \left(\Gamma^i(R)_{\mu\nu}\right)^* \Gamma^j(R)_{\alpha\beta} = \frac{h}{l_i}\delta_{ij}\delta_{\mu\alpha}\delta_{\nu\beta}$$

Great orthogonality theorem continued

$$\sum_R \left(\Gamma^i(R)_{\mu\nu}\right)^* \Gamma^j(R)_{\alpha\beta} = \frac{h}{l_i}\delta_{ij}\delta_{\mu\alpha}\delta_{\nu\beta}$$

Analysis shows that

$$\sum_i l_i^2 = h$$

Simplified analysis in terms of the "characters" of the representations

$$\chi^j(R) \equiv \sum_{\mu=1}^{l_j} \Gamma^j(R)_{\mu\mu}$$

Character orthogonality theorem

$$\sum_R \left(\chi^i(R)\right)^* \chi^j(R) = h\delta_{ij}$$

Note that all members of a class have the same character for any given representation *i.*