

## Notes on numerical solutions of Schrödinger equation

Consider the following one-dimensional Schrödinger equation:

$$\left[ -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x) \right] \psi_n(x) = E_n \psi_n(x), \quad (1)$$

where  $V(x)$  is a given potential function, and  $E_n$  is the energy eigenvalue associated with the eigenfunction  $\psi_n(x)$ . This can either represent a bound state or a continuum state. One basic approach to developing accurate numerical approximations to the solution of these equations is to use a Taylor's series expansion to relate the behavior of  $\frac{d^2\psi_n(x)}{dx^2}$  to  $\psi_n(x')$  for points  $x'$  in the neighborhood of  $x$ . Note that for any small distance  $s$ ,

$$\psi_n(x \pm s) = \psi_n(x) \pm s \frac{d\psi_n(x)}{dx} + \frac{s^2}{2} \frac{d^2\psi_n(x)}{dx^2} \pm \frac{s^3}{3} \frac{d^3\psi_n(x)}{dx^3} + \frac{s^4}{4} \frac{d^4\psi_n(x)}{dx^4} \dots \quad (2)$$

This means that if  $s$  is small, we can approximate the second derivative according to

$$\frac{d^2\psi_n(x)}{dx^2} \approx \frac{\psi_n(x+s) + \psi_n(x-s) - 2\psi_n(x)}{s^2} + O(s^4). \quad (3)$$

This central difference approximation can be used to solve both bound state and scattering state solutions of the Schrodinger equation 1. For an an example suppose the we have a bound state problem with the boundary conditions  $\psi_n(0) = \psi_n(X) = 0$  We then divide the interval  $0 \leq x \leq X$  into equal intervals with  $X = (N+1)s$  and with  $N$  interior points.

Then we can use Eq. (3) to replace the kinetic energy operator. The Schrödinger Equation then takes the form of a tri-diagonal eigenvalue problem:

$$Mv_n = \lambda_n v_n, \quad (4)$$

where

$$M = \begin{pmatrix} b_1 & c_1 & 0 & 0 & .. \\ a_2 & b_2 & c_2 & 0 & .. \\ 0 & a_3 & b_3 & c_3 & .. \\ 0 & 0 & a_4 & b_4 & .. \\ .. & .. & .. & .. & .. \end{pmatrix}. \quad (5)$$

The diagonal elements are  $b_i = 2 + s^2[2mV(is)/\hbar^2]$  and the off-diagonal elements are  $a_i \equiv c_i \equiv -1$ . Here it is assumed that  $X$  is divided into  $N$  intervals with  $X = (N+1)s$ .  $v_n$  represents a vector of  $N$  coefficients  $\{\psi_n(is)\}$ , with  $i = 1, 2, 3...N$ . The energy eigenvalues are given by  $\lambda_n = s^2[2mE_n/\hbar^2]$ . One can show that the error of this numerical procedure is of order  $O(s^4\psi^{iv}(x))$ .

By keeping the next even term in the Taylor series expansion, one can derive a Numerov algorithm for this problem which takes the form:

$$Mv_n = \lambda_n S v_n. \quad (6)$$

Here  $M$  is a tridiagonal matrix having the same form as above, and  $S$  is a positive definite tridiagonal matrix having the form:

$$S = \begin{pmatrix} \beta_1 & \gamma_1 & 0 & 0 & \dots \\ \alpha_2 & \beta_2 & \gamma_2 & 0 & \dots \\ 0 & \alpha_3 & \beta_3 & \gamma_3 & \dots \\ 0 & 0 & \alpha_4 & \beta_4 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}. \quad (7)$$

In this expression,  $\beta_i \equiv 10/12$  and  $\alpha_i \equiv \gamma_i \equiv 1/12$ , while  $b_i \equiv 2 + \frac{10}{12}s^2[2mV(is)/\hbar^2]$ ,  $a_i \equiv -1 + \frac{1}{12}s^2[2mV((i-1)s)/\hbar^2]$ , and  $c_i \equiv -1 + \frac{1}{12}s^2[2mV((i+1)s)/\hbar^2]$ . One can show that the error of this numerical procedure is of order  $O(s^6\psi^{vi}(x))$ .

For the case of a spherical atom, the wavefunction is assumed to take the form

$$\Psi_{nlm}(\mathbf{r}) = \frac{P_{nl}(r)}{r} Y_{lm}(\hat{\mathbf{r}}), \quad (8)$$

where the radial function  $p_{nl}(r)$  is determined by solving the radial Schödinger equation, which (dropping the  $nl$  indices can be written:

$$\frac{d^2 P(r)}{dr^2} = A(r)P(r), \quad (9)$$

where

$$A(r) \equiv \frac{l(l+1)}{r^2} + \frac{2m}{\hbar^2} (V(r) - E). \quad (10)$$

Rather than solving the equation in matrix form as described above, it is generally found to be more efficient to solve for each eigenvalue  $E$  iteratively, using the Numerov algorithm to integrate inward and outward and matching at an intermediate point  $r_m$ . For this purpose, we can denote  $P_i \equiv P(is)$ . The recursion formula is given by

$$P_{i+1} = \left( -\left(1 - \frac{s^2}{12}A_{i-1}\right)P_{i-1} + \left(2 + \frac{10s^2}{12}A_i\right)P_i \right) / \left(1 - \frac{s^2}{12}A_{i+1}\right). \quad (11)$$

For any given energy iteration, the correction to the energy eigenvalue can be estimated from the mismatch in the slope at the matching point:

$$\Delta E = \frac{1}{\mathcal{N}} \left( \left. \frac{dP}{dr} \right|_{\text{in}} - \left. \frac{dP}{dr} \right|_{\text{out}} \right), \quad (12)$$

where

$$\mathcal{N} \equiv \int_0^{r_m} |P(r)/P(r_m)|_{\text{out}}^2 dr + \int_{r_m}^{\infty} |P(r)/P(r_m)|_{\text{in}}^2 dr. \quad (13)$$