PHY 711 Classical Mechanics and Mathematical Methods

11-11:50 AM MWF Olin 107

Plan for Lecture 10:

Continue reading Chapter 3 & 6

1. **Constructing the Hamiltonian**
2. **Hamilton's canonical equation**
3. **Examples**

Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W Reading	Topic	Assignment	Due
1	Wed, 8/31/2016	Chap. 1	Review of basic principles	#1	9/7/2016
2	Fri, 9/02/2016	Chap. 1	Scattering theory	#2	9/7/2016
	Mon, 9/05/2016		Labor day -- no class		
3	Wed, 9/07/2016	Chap. 1	Scattering theory	#3	9/9/2016
4	Fri, 9/09/2016	Chap. 1 & 2	Scattering theory and rotations	#4	9/12/2016
5	Mon, 9/12/2016	Chap. 3	Calculus of variations	#5	9/14/2016
6	Wed, 9/14/2016	Chap. 3	Calculus of variations	#6	9/16/2016
7	Fri, 9/16/2016	Chap. 3	Lagrangian mechanics	#7	9/19/2016
8	Mon, 9/19/2016	Chap. 3 and 6	Lagrangian mechanics and constraints	#8	9/21/2016
9	Wed, 9/21/2016	Chap. 3 and 6	Constants of the motion	#9	9/23/2016
10	Fri, 9/23/2016	Chap. 3 and 6	Hamiltonian and canonical equations of motion	#10	9/30/2016
11	Mon, 9/26/2016				
12	Wed, 9/28/2016				
13	Fri, 9/30/2016				
14	Mon, 10/03/2016				

Lagrangian picture

For independent generalized coordinates $q_\sigma(t)$:

$$L = L(\{q_\sigma(t)\},\{\dot{q}_\sigma(t)\},t)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

$\Rightarrow$ Second order differential equations for $q_\sigma(t)$

Switching variables – Legendre transformation

Define: $H = H(\{q_\sigma(t)\},\{p_\sigma(t)\},t)$

$$H = \sum_\sigma \dot{q}_\sigma p_\sigma - L \qquad \text{where } p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma}$$

$$dH = \sum_\sigma \left(\dot{q}_\sigma dp_\sigma + p_\sigma d\dot{q}_\sigma - \frac{\partial L}{\partial q_\sigma} dq_\sigma - \frac{\partial L}{\partial \dot{q}_\sigma} d\dot{q}_\sigma \right) - \frac{\partial L}{\partial t} dt$$

Hamiltonian picture – continued

$$H = H(\{q_\sigma(t)\},\{p_\sigma(t)\},t)$$

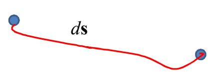
$$H = \sum_\sigma \dot{q}_\sigma p_\sigma - L \qquad \text{where} \quad p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma}$$

$$dH = \sum_\sigma \left(\dot{q}_\sigma dp_\sigma + p_\sigma d\dot{q}_\sigma - \frac{\partial L}{\partial q_\sigma} dq_\sigma - \frac{\partial L}{\partial \dot{q}_\sigma} d\dot{q}_\sigma \right) - \frac{\partial L}{\partial t} dt$$

$$= \sum_\sigma \left(\frac{\partial H}{\partial q_\sigma} dq_\sigma + \frac{\partial H}{\partial p_\sigma} dp_\sigma \right) + \frac{\partial H}{\partial t} dt$$

$$\Rightarrow \dot{q}_\sigma = \frac{\partial H}{\partial p_\sigma} \qquad \frac{\partial L}{\partial q_\sigma} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} \equiv \dot{p}_\sigma = -\frac{\partial H}{\partial q_\sigma} \qquad \frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t}$$

Direct application of Hamiltonian's principle using the Hamiltonian function --



Generalized coordinates: $q_\sigma(\{x_i\})$

Define -- Lagrangian: $L \equiv T - U$

$$L = L(\{q_\sigma\},\{\dot{q}_\sigma\},t)$$

$$\Rightarrow \text{ Minimization integral: } \quad S = \int_{t_i}^{t_f} L(\{q_\sigma\},\{\dot{q}_\sigma\},t)\,dt$$

Expressed in terms of Hamiltonian:

$$H = H(\{q_\sigma(t)\},\{p_\sigma(t)\},t)$$

$$H = \sum_\sigma \dot{q}_\sigma p_\sigma - L \qquad \Rightarrow L = \sum_\sigma \dot{q}_\sigma p_\sigma - H(\{q_\sigma(t)\},\{p_\sigma(t)\},t)$$

Hamilton's principle continued:

Minimization integral:

$$S = \int_{t_i}^{t_f} \left(\sum_\sigma \dot{q}_\sigma p_\sigma - H(\{q_\sigma(t)\},\{p_\sigma(t)\},t) \right) dt$$

$$\delta S = \int_{t_i}^{t_f} \left(\sum_\sigma \left(\dot{q}_\sigma \delta p_\sigma + \delta\dot{q}_\sigma p_\sigma - \frac{\partial H}{\partial q_\sigma}\delta q_\sigma - \frac{\partial H}{\partial p_\sigma}\delta p_\sigma \right) \right) dt = 0$$

$$\Rightarrow \dot{q}_\sigma = \frac{\partial H}{\partial p_\sigma}$$

$$\Rightarrow \dot{p}_\sigma = -\frac{\partial H}{\partial q_\sigma}$$

Canonical equations

Detail:

$$\int_{t_i}^{t_f} \left(\sum_\sigma (\delta\dot{q}_\sigma p_\sigma) \right) dt = \int_{t_i}^{t_f} \left(\sum_\sigma \left(\frac{d(\delta q_\sigma p_\sigma)}{dt} - \delta q_\sigma \dot{p}_\sigma \right) \right) dt = \cancel{\sum_\sigma \delta q_\sigma p_\sigma \Big|_{t_i}^{t_f}}^{\,0} - \int_{t_i}^{t_f} \left(\sum_\sigma (\delta q_\sigma \dot{p}_\sigma) \right) dt$$

Constants of the motion in Hamiltonian formalism

$$H = H\left(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t\right)$$

$$\frac{dq_\sigma}{dt} = \frac{\partial H}{\partial p_\sigma} \quad \Rightarrow \text{constant } q_\sigma \text{ if } \frac{\partial H}{\partial p_\sigma} = 0$$

$$\frac{dp_\sigma}{dt} = -\frac{\partial H}{\partial q_\sigma} \quad \Rightarrow \text{constant } p_\sigma \text{ if } \frac{\partial H}{\partial q_\sigma} = 0$$

$$\frac{dH}{dt} = \sum_\sigma \left(\frac{\partial H}{\partial q_\sigma} \dot{q}_\sigma + \frac{\partial H}{\partial p_\sigma} \dot{p}_\sigma \right) + \frac{\partial H}{\partial t}$$

$$\Rightarrow \text{constant } H \text{ if } \frac{\partial H}{\partial t} = 0$$

Recipe for constructing the Hamiltonian and analyzing the equations of motion

1. Construct Lagrangian function : $L = L\left(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t\right)$
2. Compute generalized momenta : $p_\sigma \equiv \frac{\partial L}{\partial \dot{q}_\sigma}$
3. Construct Hamiltonian expression : $H = \sum_\sigma \dot{q}_\sigma p_\sigma - L$
4. Form Hamiltonian function : $H = H\left(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t\right)$
5. Analyze canonical equations of motion :

$$\frac{dq_\sigma}{dt} = \frac{\partial H}{\partial p_\sigma} \qquad \frac{dp_\sigma}{dt} = -\frac{\partial H}{\partial q_\sigma}$$

Example 1 : one-dimensional potential :

$$L = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - V(z)$$

$$p_x = m\dot{x} \qquad p_y = m\dot{y} \qquad p_z = m\dot{z}$$

$$H = m\dot{x}^2 + m\dot{y}^2 + m\dot{z}^2 - \left(\tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - V(z)\right)$$

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m} + V(z)$$

Constants : p_x, p_y, H

Equations of motion : $\frac{dz}{dt} = \frac{\partial H}{\partial p_z} = \frac{p_z}{m} \qquad \frac{dp_z}{dt} = -\frac{dV}{dz}$

Example 2 : Motion in a central potential

$$L = \tfrac{1}{2} m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - V(r)$$

$$p_r = m\dot{r} \qquad p_\phi = mr^2\dot{\phi}$$

$$H = m\dot{r}^2 + mr^2\dot{\phi}^2 - \left(\tfrac{1}{2} m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - V(r)\right)$$

$$= \tfrac{1}{2} m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) + V(r)$$

$$H = \frac{p_r^2}{2m} + \frac{p_\phi^2}{2mr^2} + V(r)$$

Constants : p_ϕ, H

Equations of motion :

$$\frac{dr}{dt} = \frac{p_r}{m} \qquad \frac{dp_r}{dt} = -\frac{\partial H}{\partial r} = \frac{p_\phi^2}{mr^3} - \frac{\partial V}{\partial r}$$

Other examples

Lagrangian for symmetric top with Euler angles α, β, γ :

$$L = L(\alpha, \beta, \gamma, \dot{\alpha}, \dot{\beta}, \dot{\gamma}) = \tfrac{1}{2} I_1\left(\dot{\alpha}^2 \sin^2\beta + \dot{\beta}^2\right) + \tfrac{1}{2} I_3\left(\dot{\alpha}\cos\beta + \dot{\gamma}\right)^2 - Mgh\cos\beta$$

$$p_\alpha = I_1\dot{\alpha}\sin^2\beta + I_3\left(\dot{\alpha}\cos\beta + \dot{\gamma}\right)\cos\beta$$

$$p_\beta = I_1\dot{\beta}$$

$$p_\gamma = I_3\left(\dot{\alpha}\cos\beta + \dot{\gamma}\right)$$

$$H = \tfrac{1}{2} I_1\left(\dot{\alpha}^2 \sin^2\beta + \dot{\beta}^2\right) + \tfrac{1}{2} I_3\left(\dot{\alpha}\cos\beta + \dot{\gamma}\right)^2 + Mgh\cos\beta$$

$$H = \frac{\left(p_\alpha - p_\gamma\cos\beta\right)^2}{2I_1\sin^2\beta} + \frac{p_\beta^2}{2I_1} + \frac{p_\gamma^2}{2I_3} + Mgh\cos\beta$$

Constants : p_α, p_γ, H

Other examples

$$L = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + \frac{q}{2c} B_0\left(-\dot{x}y + \dot{y}x\right)$$

$$p_x = m\dot{x} - \frac{q}{2c} B_0 y$$

$$p_y = m\dot{y} + \frac{q}{2c} B_0 x$$

$$p_z = m\dot{z}$$

$$H = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$$

$$H = \frac{\left(p_x + \frac{q}{2c} B_0 y\right)^2}{2m} + \frac{\left(p_y - \frac{q}{2c} B_0 x\right)^2}{2m} + \frac{p_z^2}{2m}$$

Constants : p_z, H

Canonical equations of motion for constant magnetic field:

$$H = \frac{\left(p_x + \frac{q}{2c}B_0 y\right)^2}{2m} + \frac{\left(p_y - \frac{q}{2c}B_0 x\right)^2}{2m} + \frac{p_z^2}{2m}$$

Constants : $\quad p_z, H$

$$\frac{dx}{dt} = \frac{p_x + \frac{q}{2c}B_0 y}{m} \qquad \frac{dy}{dt} = \frac{p_y - \frac{q}{2c}B_0 x}{m}$$

$$\frac{dp_x}{dt} = -\frac{\partial H}{\partial x} = \frac{qB_0}{2mc}\left(p_y - \frac{q}{2c}B_0 x\right)$$

$$\frac{dp_y}{dt} = -\frac{\partial H}{\partial y} = -\frac{qB_0}{2mc}\left(p_x + \frac{q}{2c}B_0 y\right)$$

Canonical equations of motion for constant magnetic field -- continued:

$$\frac{dx}{dt} = \frac{p_x + \frac{q}{2c}B_0 y}{m} \qquad \frac{dy}{dt} = \frac{p_y - \frac{q}{2c}B_0 x}{m}$$

$$\frac{dp_x}{dt} = \frac{qB_0}{2mc}\left(p_y - \frac{q}{2c}B_0 x\right) = \frac{qB_0}{2c}\frac{dy}{dt}$$

$$\frac{dp_y}{dt} = -\frac{qB_0}{2mc}\left(p_x + \frac{q}{2c}B_0 y\right) = -\frac{qB_0}{2c}\frac{dx}{dt}$$

$$\frac{d^2x}{dt^2} = \frac{\dot{p}_x}{m} + \frac{q}{2mc}B_0\dot{y} = \frac{qB_0}{mc}\frac{dy}{dt}$$

$$\frac{d^2y}{dt^2} = \frac{\dot{p}_y}{m} - \frac{q}{2mc}B_0\dot{x} = -\frac{qB_0}{mc}\frac{dx}{dt}$$

General treatment of particle of mass m and charge q moving in 3 dimensions in an potential $U(\mathbf{r})$ as well as electromagnetic scalar and vector potentials $\Phi(\mathbf{r},t)$ and $\mathbf{A}(\mathbf{r},t)$:

Lagrangian: $\quad L(\mathbf{r},\dot{\mathbf{r}},t) = \frac{1}{2}m\dot{\mathbf{r}}^2 - U(\mathbf{r}) - q\Phi(\mathbf{r},t) + \frac{q}{c}\dot{\mathbf{r}}\cdot\mathbf{A}(\mathbf{r},t)$

Hamiltonian: $\quad \mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{r}}} = m\dot{\mathbf{r}} + \frac{q}{c}\mathbf{A}(\mathbf{r},t)$

$$H(\mathbf{r},\mathbf{p},t) = \mathbf{p}\cdot\dot{\mathbf{r}} - L(\mathbf{r},\dot{\mathbf{r}},t)$$

$$= \frac{1}{2m}\left(\mathbf{p} - \frac{q}{c}\mathbf{A}(\mathbf{r},t)\right)^2 + U(\mathbf{r}) + q\Phi(\mathbf{r},t)$$

Poisson brackets:
Recall:

$$H = H(\{q_\sigma(t)\},\{p_\sigma(t)\},t)$$

$$\frac{dq_\sigma}{dt} = \frac{\partial H}{\partial p_\sigma} \quad \Rightarrow \text{constant } q_\sigma \text{ if } \frac{\partial H}{\partial p_\sigma} = 0$$

$$\frac{dp_\sigma}{dt} = -\frac{\partial H}{\partial q_\sigma} \quad \Rightarrow \text{constant } p_\sigma \text{ if } \frac{\partial H}{\partial q_\sigma} = 0$$

$$\frac{dH}{dt} = \sum_\sigma \left(\frac{\partial H}{\partial q_\sigma}\dot{q}_\sigma + \frac{\partial H}{\partial p_\sigma}\dot{p}_\sigma \right) + \frac{\partial H}{\partial t}$$

Similarly for an arbitrary function : $F = F(\{q_\sigma(t)\},\{p_\sigma(t)\},t)$

$$\frac{dF}{dt} = \sum_\sigma \left(\frac{\partial F}{\partial q_\sigma}\dot{q}_\sigma + \frac{\partial F}{\partial p_\sigma}\dot{p}_\sigma \right) + \frac{\partial F}{\partial t} = \sum_\sigma \left(\frac{\partial F}{\partial q_\sigma}\frac{\partial H}{\partial p_\sigma} - \frac{\partial F}{\partial p_\sigma}\frac{\partial H}{\partial q_\sigma} \right) + \frac{\partial F}{\partial t}$$

Poisson brackets -- continued:

For an arbitrary function : $F = F(\{q_\sigma(t)\},\{p_\sigma(t)\},t)$

$$\frac{dF}{dt} = \sum_\sigma \left(\frac{\partial F}{\partial q_\sigma}\dot{q}_\sigma + \frac{\partial F}{\partial p_\sigma}\dot{p}_\sigma \right) + \frac{\partial F}{\partial t} = \sum_\sigma \left(\frac{\partial F}{\partial q_\sigma}\frac{\partial H}{\partial p_\sigma} - \frac{\partial F}{\partial p_\sigma}\frac{\partial H}{\partial q_\sigma} \right) + \frac{\partial F}{\partial t}$$

Define :

$$[F,G]_{PB} \equiv \sum_\sigma \left(\frac{\partial F}{\partial q_\sigma}\frac{\partial G}{\partial p_\sigma} - \frac{\partial F}{\partial p_\sigma}\frac{\partial G}{\partial q_\sigma} \right) = -[G,F]_{PB}$$

So that : $\frac{dF}{dt} = [F,H]_{PB} + \frac{\partial F}{\partial t}$

Poisson brackets -- continued:

$$[F,G]_{PB} \equiv \sum_\sigma \left(\frac{\partial F}{\partial q_\sigma}\frac{\partial G}{\partial p_\sigma} - \frac{\partial F}{\partial p_\sigma}\frac{\partial G}{\partial q_\sigma} \right) = -[G,F]_{PB}$$

Examples :

$[x,x]_{PB} = 0 \qquad [x,p_x]_{PB} = 1 \qquad [x,p_y]_{PB} = 0$

$[L_x,L_y]_{PB} = L_z$

Liouville theorem

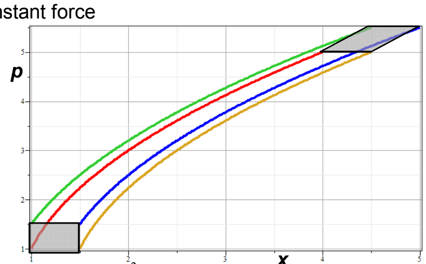
Let $D \equiv$ density of particles in phase space :

$$\frac{dD}{dt} = [D,H]_{PB} + \frac{\partial D}{\partial t} = 0$$

Phase space

Phase space is defined at the set of all coordinates and momenta of a system :
$(\{q_\sigma(t)\}, \{p_\sigma(t)\})$
For a d dimensional system with N particles, the phase space corresponds to $2dN$ degrees of freedom.

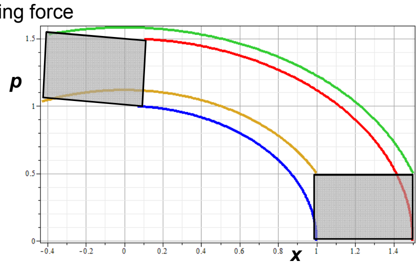
Phase space diagram for one-dimensional motion due to constant force



$$H(x,p) = \frac{p^2}{2m} - F_0 x \qquad \dot{p} = F_0 \qquad \dot{x} = \frac{p}{m}$$

$$p_i(t) = p_{0i} + F_0 t \qquad x_i(t) = x_{0i} + \frac{p_{0i}}{m} t + \frac{1}{2} F_0 t^2$$

Phase space diagram for one-dimensional motion due to spring force



$$H(x,p) = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2 \qquad \dot{p} = -m\omega^2 x \qquad \dot{x} = \frac{p}{m}$$

$$p_i(t) = p_{0i} \cos(\omega t + \theta_{0i}) \qquad x_i(t) = \frac{p_{0i}}{m\omega} \sin(\omega t + \theta_{0i})$$

Liouville's Theorem (1838)

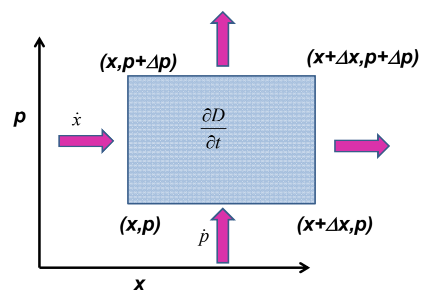
The density of representative points in phase space corresponding to the motion of a system of particles remains constant during the motion.

Denote the density of particles in phase space: $D = D(\{q_\sigma(t)\},\{p_\sigma(t)\},t)$

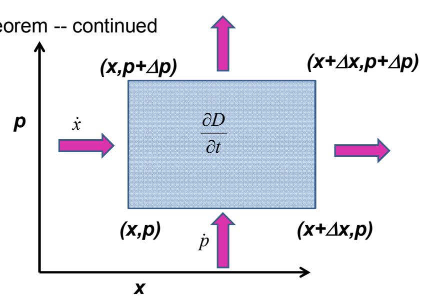
$$\frac{dD}{dt} = \sum_\sigma \left(\frac{\partial D}{\partial q_\sigma}\dot{q}_\sigma + \frac{\partial D}{\partial p_\sigma}\dot{p}_\sigma \right) + \frac{\partial D}{\partial t}$$

According to Liouville's theorem: $\frac{dD}{dt} = 0$

Liouville's theorem



Liouville's theorem -- continued

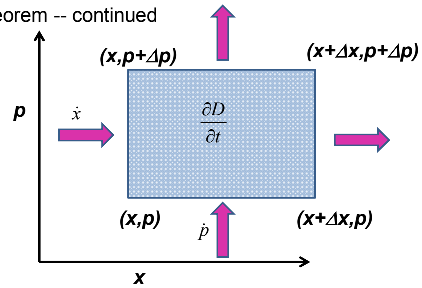


$\frac{\partial D}{\partial t} \Rightarrow$ time rate of change of particles within volume

= time rate of particle entering minus particles leaving

$$= -\frac{\partial D}{\partial x}\dot{x} - \frac{\partial D}{\partial p}\dot{p}$$

Liouville's theorem -- continued



$$\frac{\partial D}{\partial t} = -\frac{\partial D}{\partial x}\dot{x} - \frac{\partial D}{\partial p}\dot{p}$$

$$\frac{\partial D}{\partial t} + \frac{\partial D}{\partial x}\dot{x} + \frac{\partial D}{\partial p}\dot{p} = 0 = \frac{dD}{dt}$$

Review:
Liouville's theorem:
Imagine a collection of particles obeying the Canonical equations of motion in phase space.

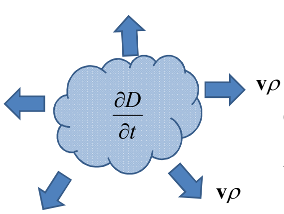
Let D denote the "distribution" of particles in phase space:

$$D = D(\{q_1 \cdots q_{3N}\}, \{p_1 \cdots p_{3N}\}, t)$$

Liouville's theorm shows that:

$$\frac{dD}{dt} = 0 \qquad \Rightarrow D \text{ is constant in time}$$

Proof of Liouville'e theorem:



Continuity equation:

$$\frac{\partial D}{\partial t} = -\nabla \cdot (\mathbf{v}D)$$

Note: in this case, the velocity is the $6N$ dimensional vector:

$$\mathbf{v} = (\dot{\mathbf{r}}_1, \dot{\mathbf{r}}_2, \ldots \dot{\mathbf{r}}_N, \dot{\mathbf{p}}_1, \dot{\mathbf{p}}_2, \ldots \dot{\mathbf{p}}_N)$$

We also have a $6N$ dimensional gradient:

$$\nabla = (\nabla_{\mathbf{r}_1}, \nabla_{\mathbf{r}_2}, \ldots \nabla_{\mathbf{r}_N}, \nabla_{\mathbf{p}_1}, \nabla_{\mathbf{p}_2}, \ldots \nabla_{\mathbf{p}_N})$$

$$\frac{\partial D}{\partial t} = -\nabla \cdot (\mathbf{v}D)$$
$$= -\sum_{j=1}^{3N}\left[\frac{\partial}{\partial q_j}(\dot{q}_j D) + \frac{\partial}{\partial p_j}(\dot{p}_j D)\right]$$
$$= -\sum_{j=1}^{3N}\left[\frac{\partial D}{\partial q_j}\dot{q}_j + \frac{\partial D}{\partial p_j}\dot{p}_j\right] - D\sum_{j=1}^{3N}\left[\frac{\partial \dot{q}_j}{\partial q_j} + \frac{\partial \dot{p}_j}{\partial p_j}\right]$$
$$\frac{\partial \dot{q}_j}{\partial q_j} + \frac{\partial \dot{p}_j}{\partial p_j} = \frac{\partial^2 H}{\partial q_j \partial p_j} + \left(-\frac{\partial^2 H}{\partial p_j \partial q_j}\right) = 0$$

$$\frac{\partial D}{\partial t} = -\sum_{j=1}^{3N}\left[\frac{\partial D}{\partial q_j}\dot{q}_j + \frac{\partial D}{\partial p_j}\dot{p}_j\right] - D\cancelto{0}{\sum_{j=1}^{3N}\left[\frac{\partial \dot{q}_j}{\partial q_j} + \frac{\partial \dot{p}_j}{\partial p_j}\right]}$$

$$\frac{\partial D}{\partial t} = -\sum_{j=1}^{3N}\left[\frac{\partial D}{\partial q_j}\dot{q}_j + \frac{\partial D}{\partial p_j}\dot{p}_j\right]$$
$$\Rightarrow \frac{\partial D}{\partial t} + \sum_{j=1}^{3N}\left[\frac{\partial D}{\partial q_j}\dot{q}_j + \frac{\partial D}{\partial p_j}\dot{p}_j\right] = \frac{dD}{dt} = 0$$

$$\boxed{\frac{dD}{dt} = 0}$$

Importance of Liouville's theorem to statistical mechanical analysis:

In statistical mechanics, we need to evaluate the probability of various configurations of particles. The fact that the density of particles in phase space is constant in time, implies that each point in phase space is equally probable and that the time average of the evolution of a system can be determined by an average of the system over phase space volume.