PHY 711 Classical Mechanics and Mathematical Methods
11-11:50 AM MWF Olin 107

Plan for Lecture 12:

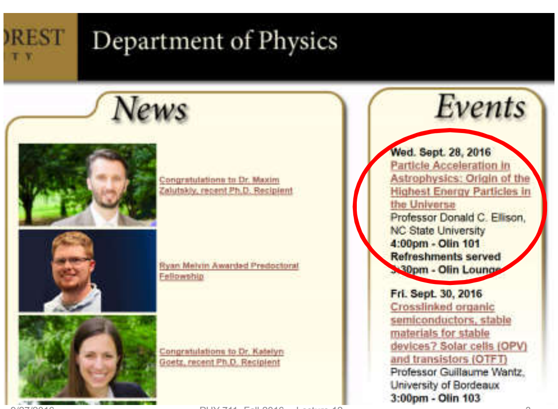
Finish reading Chapter 6

1. Virial theorem
2. Canonical transformations
3. Hamilton-Jacobi formalism

Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W Reading	Topic	Assignment	Due
1	Wed, 8/31/2016	Chap. 1	Review of basic principles	#1	9/7/2016
2	Fri, 9/02/2016	Chap. 1	Scattering theory	#2	9/7/2016
	Mon, 9/05/2016		Labor day -- no class		
3	Wed, 9/07/2016	Chap. 1	Scattering theory	#3	9/9/2016
4	Fri, 9/09/2016	Chap. 1 & 2	Scattering theory and rotations	#4	9/12/2016
5	Mon, 9/12/2016	Chap. 3	Calculus of variations	#5	9/14/2016
6	Wed, 9/14/2016	Chap. 3	Calculus of variations	#6	9/16/2016
7	Fri, 9/16/2016	Chap. 3	Lagrangian mechanics	#7	9/19/2016
8	Mon, 9/19/2016	Chap. 3 and 6	Lagrangian mechanics and constraints	#8	9/21/2016
9	Wed, 9/21/2016	Chap. 3 and 6	Constants of the motion	#9	9/23/2016
10	Fri, 9/23/2016	Chap. 3 and 6	Hamiltonian and canonical equations of motion	#10	9/26/2016
11	Mon, 9/26/2016	Chap. 3 and 6	Phase space	#11	9/28/2016
12	Wed, 9/28/2016	Chap. 6	Canonical transformations	#12	9/30/2016
13	Fri, 9/30/2016				
14	Mon, 10/03/2016				



WFU Physics Colloquium

TITLE: Particle Acceleration in Astrophysics: Origin of the Highest Energy Particles in the Universe

SPEAKER: Professor Donald C. Ellison,

Department of Physics
North Carolina State University

TIME: Wednesday September 28, 2016 at 4:00 PM

PLACE: Room 101 Olin Physical Laboratory

Refreshments will be served at 3:30 PM in the Olin Lounge. All interested persons are cordially invited to attend.

ABSTRACT

Single cosmic ray protons have been measured with energies above 2 x 10^{20} eV. This is equivalent to a baseball moving at 50 mph and energies this high haven't existed elsewhere since the temperature of the Universe dropped below ~10^{24} K, about 10^{-28} seconds after the big bang. I will mention the connections cosmic rays provide between astronomy, plasma physics, and particle physics, and describe briefly why cosmic ray origin is still a mystery more than 100 years after their discovery. Collisionless shocks exist throughout space on all scales and these shocks are known to be efficient particle accelerators. Supernova remnant blast waves show the clearest evidence for producing cosmic rays to extreme energies and the broadband radiation observed from individual SNRs gives the most complete picture of particle acceleration available. However, the high efficiency inferred for Fermi shock acceleration means nonlinear effects are important and these complex problems are actively being studied.

Virial theorem (Clausius ~ 1860)

$$2\langle T\rangle = -\left\langle \sum_\sigma \mathbf{F}_\sigma \cdot \mathbf{r}_\sigma \right\rangle$$

Proof:

Define: $A \equiv \sum_\sigma \mathbf{p}_\sigma \cdot \mathbf{r}_\sigma$

$$\frac{dA}{dt} = \sum_\sigma (\dot{\mathbf{p}}_\sigma \cdot \mathbf{r}_\sigma + \mathbf{p}_\sigma \cdot \dot{\mathbf{r}}_\sigma) = \sum_\sigma \mathbf{F}_\sigma \cdot \mathbf{r} + 2T$$

Because $\dot{\mathbf{p}}_\sigma = \mathbf{F}_\sigma$

$$\left\langle \frac{dA}{dt} \right\rangle = \left\langle \sum_\sigma \mathbf{F}_\sigma \cdot \mathbf{r} \right\rangle + 2\langle T\rangle$$

$$\left\langle \frac{dA}{dt} \right\rangle = \frac{1}{\tau}\int_0^\tau \frac{dA(t)}{dt}dt = \frac{A(\tau)-A(0)}{\tau} \Rightarrow 0$$

Note that this implies that the motion is bounded

$$\Rightarrow \left\langle \sum_\sigma \mathbf{F}_\sigma \cdot \mathbf{r} \right\rangle + 2\langle T\rangle = 0$$

Examples of the Virial Theorem $2\langle T\rangle = -\left\langle \sum_\sigma \mathbf{F}_\sigma \cdot \mathbf{r}_\sigma \right\rangle$

Harmonic oscillator:

$\mathbf{F} = -kx\hat{\mathbf{x}}$ $\quad T = \frac{1}{2}m\dot{x}^2$ $\quad \langle m\dot{x}^2\rangle = \langle kx^2\rangle$

Check: for $x(t) = A\sin\left(\sqrt{\frac{k}{m}}t + \alpha\right)$

$$\langle m\dot{x}^2\rangle = \frac{1}{2}kA^2 = \langle kx^2\rangle$$

Examples of the Virial Theorem $2\langle T\rangle = -\left\langle \sum_\sigma \mathbf{F}_\sigma \cdot \mathbf{r}_\sigma \right\rangle$

Circular orbit due to gravitational field of massive object:

$$\mathbf{F} = -\frac{GMm}{r^2}\hat{\mathbf{r}} \qquad T = \frac{1}{2}mr^2\omega^2 \qquad \langle mr^2\omega^2\rangle = \left\langle \frac{GMm}{r}\right\rangle$$

Check: for $r\omega^2 = \frac{GM}{r^2}$ $\Rightarrow \langle mr^2\omega^2\rangle = \left\langle \frac{GMm}{r}\right\rangle$

Hamiltonian formalism and the canonical equations of motion:

$$H = H(\{q_\sigma(t)\},\{p_\sigma(t)\},t)$$

Canonical equations of motion

$$\frac{dq_\sigma}{dt} = \frac{\partial H}{\partial p_\sigma}$$

$$\frac{dp_\sigma}{dt} = -\frac{\partial H}{\partial q_\sigma}$$

Notion of "Canonical" transformations

$$q_\sigma = q_\sigma(\{Q_1\cdots Q_n\},\{P_1\cdots P_n\},t) \qquad \text{for each } \sigma$$

$$p_\sigma = p_\sigma(\{Q_1\cdots Q_n\},\{P_1\cdots P_n\},t) \qquad \text{for each } \sigma$$

For some $\tilde{H}$ and F, using Legendre transformations

$$\sum_\sigma p_\sigma \dot{q}_\sigma - H(\{q_\sigma\},\{p_\sigma\},t) = \sum_\sigma P_\sigma \dot{Q}_\sigma - \tilde{H}(\{Q_\sigma\},\{P_\sigma\},t) + \frac{d}{dt}F(\{q_\sigma\},\{Q_\sigma\},t)$$

Apply Hamilton's principle:

$$\delta \int_{t_i}^{t_f}\left[\sum_\sigma P_\sigma \dot{Q}_\sigma - \tilde{H}(\{Q_\sigma\},\{P_\sigma\},t) + \frac{d}{dt}F(\{q_\sigma\},\{Q_\sigma\},t)\right]dt = 0$$

$$\delta \int_{t_i}^{t_f}\left[\frac{d}{dt}F(\{q_\sigma\},\{Q_\sigma\},t)\right]dt = \int_{t_i}^{t_f}\left[\frac{d}{dt}\delta F(\{q_\sigma\},\{Q_\sigma\},t)\right]dt$$

$$= \delta F(t_f) - \delta F(t_i) = 0 \qquad \text{and} \qquad \dot{Q}_\sigma = \frac{\partial \tilde{H}}{\partial P_\sigma} \qquad \dot{P}_\sigma = -\frac{\partial \tilde{H}}{\partial Q_\sigma}$$

Some relations between old and new variables:

$$\sum_\sigma p_\sigma \dot{q}_\sigma - H(\{q_\sigma\},\{p_\sigma\},t) =$$

$$\sum_\sigma P_\sigma \dot{Q}_\sigma - \widetilde{H}(\{Q_\sigma\},\{P_\sigma\},t) + \frac{d}{dt}F(\{q_\sigma\},\{Q_\sigma\},t)$$

$$\frac{d}{dt}F(\{q_\sigma\},\{Q_\sigma\},t) = \sum_\sigma \left(\left(\frac{\partial F}{\partial q_\sigma}\right)\dot{q}_\sigma + \left(\frac{\partial F}{\partial Q_\sigma}\right)\dot{Q}_\sigma \right) + \frac{\partial F}{\partial t}$$

$$\Rightarrow \sum_\sigma \left(p_\sigma - \left(\frac{\partial F}{\partial q_\sigma}\right)\right)\dot{q}_\sigma - H(\{q_\sigma\},\{p_\sigma\},t) =$$

$$\sum_\sigma \left(P_\sigma + \left(\frac{\partial F}{\partial Q_\sigma}\right)\right)\dot{Q}_\sigma - \tilde{H}(\{Q_\sigma\},\{P_\sigma\},t) + \frac{\partial F}{\partial t}$$

$$\sum_\sigma \left(p_\sigma - \left(\frac{\partial F}{\partial q_\sigma}\right)\right)\dot{q}_\sigma - H(\{q_\sigma\},\{p_\sigma\},t) =$$

$$\sum_\sigma \left(P_\sigma + \left(\frac{\partial F}{\partial Q_\sigma}\right)\right)\dot{Q}_\sigma - \widetilde{H}(\{Q_\sigma\},\{P_\sigma\},t) + \frac{\partial F}{\partial t}$$

$$\Rightarrow p_\sigma = \left(\frac{\partial F}{\partial q_\sigma}\right) \qquad P_\sigma = -\left(\frac{\partial F}{\partial Q_\sigma}\right)$$

$$\Rightarrow \widetilde{H}(\{Q_\sigma\},\{P_\sigma\},t) = H(\{q_\sigma\},\{p_\sigma\},t) + \frac{\partial F}{\partial t}$$

Note that it is conceivable that if we were extraordinarily clever, we could find all of the constants of the motion!

$$\dot{Q}_\sigma = \frac{\partial \widetilde{H}}{\partial P_\sigma} \qquad \dot{P}_\sigma = -\frac{\partial \widetilde{H}}{\partial Q_\sigma}$$

$$\text{Suppose: } \dot{Q}_\sigma = \frac{\partial \widetilde{H}}{\partial P_\sigma} = 0 \text{ and } \dot{P}_\sigma = -\frac{\partial \widetilde{H}}{\partial Q_\sigma} = 0$$

$$\Rightarrow Q_\sigma, P_\sigma \quad \text{are constants of the motion}$$

Possible solution – Hamilton-Jacobi theory:

$$\text{Suppose: } \quad F(\{q_\sigma\},\{Q_\sigma\},t) \Rightarrow -\sum_\sigma P_\sigma Q_\sigma + S(\{q_\sigma\},\{P_\sigma\},t)$$

$$\sum_\sigma p_\sigma \dot{q}_\sigma - H(\{q_\sigma\},\{p_\sigma\},t) =$$

$$\sum_\sigma P_\sigma \dot{Q}_\sigma - \tilde{H}(\{Q_\sigma\},\{P_\sigma\},t) + \frac{d}{dt}\left(-\sum_\sigma P_\sigma Q_\sigma + S(\{q_\sigma\},\{P_\sigma\},t)\right)$$

$$= -\tilde{H}(\{Q_\sigma\},\{P_\sigma\},t) - \sum_\sigma \dot{P}_\sigma Q_\sigma + \sum_\sigma\left(\frac{\partial S}{\partial q_\sigma}\dot{q}_\sigma + \frac{\partial S}{\partial P_\sigma}\dot{P}_\sigma\right) + \frac{\partial S}{\partial t}$$

Solution :

$$p_\sigma = \frac{\partial S}{\partial q_\sigma} \qquad Q_\sigma = \frac{\partial S}{\partial P_\sigma}$$

$$\tilde{H}(\{Q_\sigma\},\{P_\sigma\},t) = H(\{q_\sigma\},\{p_\sigma\},t) + \frac{\partial S}{\partial t}$$

When the dust clears :

Assume $\{Q_\sigma\},\{P_\sigma\},\tilde{H}$ are constants; choose $\tilde{H} = 0$

Need to find $S(\{q_\sigma\},\{P_\sigma\},t)$

$$p_\sigma = \frac{\partial S}{\partial q_\sigma} \qquad Q_\sigma = \frac{\partial S}{\partial P_\sigma}$$

$$\Rightarrow H\left(\{q_\sigma\},\left\{\frac{\partial S}{\partial q_\sigma}\right\},t\right) + \frac{\partial S}{\partial t} = 0$$

Note : S is the "action":

$$\sum_\sigma p_\sigma \dot{q}_\sigma - H(\{q_\sigma\},\{p_\sigma\},t) =$$

$$\underbrace{\sum_\sigma P_\sigma \dot{Q}_\sigma}_{0} - \underbrace{\tilde{H}(\{Q_\sigma\},\{P_\sigma\},t)}_{0} + \frac{d}{dt}\left(-\underbrace{\sum_\sigma P_\sigma Q_\sigma}_{0} + S(\{q_\sigma\},\{P_\sigma\},t)\right)$$

$$\sum_\sigma p_\sigma \dot{q}_\sigma - H(\{q_\sigma\},\{p_\sigma\},t) =$$

$$\underbrace{\sum_\sigma P_\sigma \dot{Q}_\sigma}_{0} - \underbrace{\tilde{H}(\{Q_\sigma\},\{P_\sigma\},t)}_{0} + \frac{d}{dt}\left(-\underbrace{\sum_\sigma P_\sigma Q_\sigma}_{0} + S(\{q_\sigma\},\{P_\sigma\},t)\right)$$

$$\int_{t_i}^{t_f}\left(\sum_\sigma p_\sigma \dot{q}_\sigma - H(\{q_\sigma\},\{p_\sigma\},t)\right)dt = \int_{t_i}^{t_f}\left(\frac{d}{dt}\left(S(\{q_\sigma\},\{P_\sigma\},t)\right)\right)dt$$

$$= S(\{q_\sigma\},\{P_\sigma\},t)\Big|_{t_i}^{t_f}$$

Differential equation for ***S***:

$$H\left(\{q_\sigma\},\left\{\frac{\partial S}{\partial q_\sigma}\right\},t\right)+\frac{\partial S}{\partial t}=0$$

Example: $\quad H(\{q\},\{p\},t)=\frac{p^2}{2m}+\frac{1}{2}m\omega^2q^2$

Hamilton - Jacobi Eq : $\quad H\left(\{q\},\left\{\frac{\partial S}{\partial q}\right\},t\right)+\frac{\partial S}{\partial t}=0$

$$\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2+\frac{1}{2}m\omega^2q^2+\frac{\partial S}{\partial t}=0$$

Assume: $\quad S(q,t)\equiv W(q)-Et \qquad (E \quad \text{constant})$

Continued:

$$\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2+\frac{1}{2}m\omega^2q^2+\frac{\partial S}{\partial t}=0$$

Assume: $\quad S(q,t)\equiv W(q)-Et \qquad (E \quad \text{constant})$

$$\frac{1}{2m}\left(\frac{dW}{dq}\right)^2+\frac{1}{2}m\omega^2q^2=E$$

$$\frac{dW}{dq}=\sqrt{2mE-(m\omega)^2q^2}$$

$$W(q)=\int\sqrt{2mE-(m\omega)^2q^2}\,dq$$

Continued:

$$W(q)=\int\sqrt{2mE-(m\omega)^2q^2}\,dq$$

$$=\frac{1}{2}q\sqrt{2mE-(m\omega)^2q^2}+\frac{E}{\omega}\sin^{-1}\left(\frac{m\omega q}{\sqrt{2mE}}\right)+C$$

$$S(q,E,t)=\frac{1}{2}q\sqrt{2mE-(m\omega)^2q^2}+\frac{E}{\omega}\sin^{-1}\left(\frac{m\omega q}{\sqrt{2mE}}\right)-Et$$

$$\frac{\partial S}{\partial E}=Q=\frac{1}{\omega}\sin^{-1}\left(\frac{m\omega q}{\sqrt{2mE}}\right)-t$$

$$\Rightarrow q(t)=\frac{\sqrt{2mE}}{m\omega}\sin(\omega(t+Q))$$

Another example of Hamilton Jacobi equations

Example: $H\left(\{q\},\{p\},t\right)=\frac{p^2}{2m}+mgy$

Assume $y(0)=h; \quad p(0)=0$

Hamilton-Jacobi Eq: $H\left(\{q\},\left\{\frac{\partial S}{\partial q}\right\},t\right)+\frac{\partial S}{\partial t}=0$

$$\frac{1}{2m}\left(\frac{\partial S}{\partial y}\right)^2+mgy+\frac{\partial S}{\partial t}=0$$

Assume: $S(y,t)\equiv W(y)-Et \qquad (E \text{ constant})$

Example: $H\left(\{q\},\{p\},t\right)=\frac{p^2}{2m}+mgy$

Assume $y(0)=h; \quad p(0)=0$

$$\frac{1}{2m}\left(\frac{dW}{dy}\right)^2+mgy=E\equiv mgh$$

$$W(y)=m\int_y^h\sqrt{2g\left(h-y'\right)}dy'=\frac{2}{3}m\sqrt{2g}\left(h-y\right)^{3/2}$$

$$S(y,t)=W(y)-Et=\frac{2}{3}m\sqrt{2g}\left(h-y\right)^{3/2}-mght$$

Check action:

For this case: $y(t)=h-\frac{1}{2}gt^2$

$$S=\int_0^t\left(\frac{1}{2}m\dot{y}^2-mgy\right)dt'=\frac{1}{3}mg^2t^3-mght$$

$$S(y,t)=W(y)-Et=\frac{2}{3}m\sqrt{2g}\left(h-y\right)^{3/2}-mght$$

Recap --

Lagrangian picture

For independent generalized coordinates $q_\sigma(t)$:

$$L = L(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

$\Rightarrow$ Second order differential equations for $q_\sigma(t)$

Hamiltonian picture

$$H = H(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t)$$

$$\frac{dq_\sigma}{dt} = \frac{\partial H}{\partial p_\sigma} \qquad \frac{dp_\sigma}{dt} = -\frac{\partial H}{\partial q_\sigma}$$

$\Rightarrow$ Coupled first order differential equations for

$q_\sigma(t)$ and $p_\sigma(t)$