

**PHY 711 Classical Mechanics and
Mathematical Methods
11-11:50 AM MWF Olin 107**

Plan for Lecture 13:

Start reading Chapter 4

**1. Small oscillations about
equilibrium**

2. Normal modes

9/30/2016

PHY 711 Fall 2016 -- Lecture 13

1

Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

Date	F&W Reading Topic	Assignment Due
1 Wed, 8/31/2016	Chap. 1 Review of basic principles	#1 9/7/2016
2 Fri, 9/02/2016	Chap. 1 Scattering theory	#2 9/7/2016
Mon, 9/05/2016	Labor day -- no class	
3 Wed, 9/07/2016	Chap. 1 Scattering theory	#3 9/9/2016
4 Fri, 9/09/2016	Chap. 1 & 2 Scattering theory and rotations	#4 9/12/2016
5 Mon, 9/12/2016	Chap. 3 Calculus of variations	#5 9/14/2016
6 Wed, 9/14/2016	Chap. 3 Calculus of variations	#6 9/16/2016
7 Fri, 9/16/2016	Chap. 3 Lagrangian mechanics	#7 9/19/2016
8 Mon, 9/19/2016	Chap. 3 and 6 Lagrangian mechanics and constraints	#8 9/21/2016
9 Wed, 9/21/2016	Chap. 3 and 6 Constants of the motion	#9 9/23/2016
10 Fri, 9/23/2016	Chap. 3 and 6 Hamiltonian and canonical equations of motion	#10 9/26/2016
11 Mon, 9/26/2016	Chap. 3 and 6 Phase space	#11 9/28/2016
12 Wed, 9/28/2016	Chap. 6 Canonical transformations	#12 9/30/2016
13 Fri, 9/30/2016	Chap. 4 Small oscillations	#13 10/04/2016
14 Mon, 10/03/2016		
15 Wed, 10/05/2016		

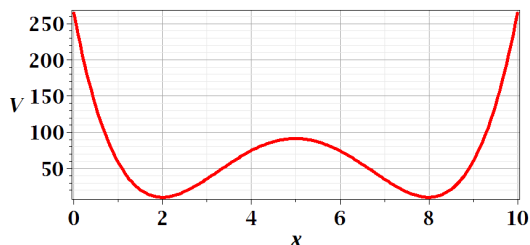
9/30/2016

PHY 711 Fall 2016 -- Lecture 13

2

Motivation for studying small oscillations -- many interacting systems have stable and meta-stable configurations which are well approximated by:

$$V(x) \approx V(x_{eq}) + \frac{1}{2}(x - x_{eq})^2 \left. \frac{d^2V}{dx^2} \right|_{x_{eq}} = V(x_{eq}) + \frac{1}{2}k(x - x_{eq})^2$$



9/30/2016

PHY 711 Fall 2016 -- Lecture 13

3

Equations of motion for a single oscillator:

Let $k \equiv m\omega^2$

$$L(x, \dot{x}, t) = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x} \quad \Rightarrow \quad m\ddot{x} = -m\omega^2 x$$

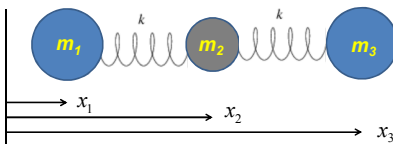
$$x(t) = A\sin(\omega t + \varphi)$$

9/30/2016

PHY 711 Fall 2016 -- Lecture 13

4

Example – linear molecule



$$L = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}m_3\dot{x}_3^2 - \frac{1}{2}k(x_2 - x_1 - \ell_{12})^2 - \frac{1}{2}k(x_3 - x_2 - \ell_{23})^2$$

9/30/2016

PHY 711 Fall 2016 -- Lecture 13

5

$$L = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}m_3\dot{x}_3^2 - \frac{1}{2}k(x_2 - x_1 - \ell_{12})^2 - \frac{1}{2}k(x_3 - x_2 - \ell_{23})^2$$

Let: $x_1 \rightarrow x_1 - x_1^0$ $x_2 \rightarrow x_2 - x_1^0 - \ell_{12}$ $x_3 \rightarrow x_3 - x_1^0 - \ell_{12} - \ell_{23}$

$$L = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}m_3\dot{x}_3^2 - \frac{1}{2}k(x_2 - x_1)^2 - \frac{1}{2}k(x_3 - x_2)^2$$

Coupled equations of motion :

$$m_1\ddot{x}_1 = k(x_2 - x_1)$$

$$m_2\ddot{x}_2 = -k(x_2 - x_1) + k(x_3 - x_2) = k(x_1 - 2x_2 + x_3)$$

$$m_3\ddot{x}_3 = -k(x_3 - x_2)$$

9/30/2016

PHY 711 Fall 2016 -- Lecture 13

6

Coupled equations of motion :

$$m_1 \ddot{x}_1 = k(x_2 - x_1)$$

$$m_2 \ddot{x}_2 = -k(x_2 - x_1) + k(x_3 - x_2) = k(x_1 - 2x_2 + x_3)$$

$$m_3 \ddot{x}_3 = -k(x_3 - x_2)$$

Let $x_i(t) = X_i^\alpha e^{i\omega_\alpha t}$

$$-\omega_\alpha^2 m_1 X_1^\alpha = k(X_2^\alpha - X_1^\alpha)$$

$$-\omega_\alpha^2 m_2 X_2^\alpha = k(X_1^\alpha - 2X_2^\alpha + X_3^\alpha)$$

$$-\omega_\alpha^2 m_3 X_3^\alpha = -k(X_3^\alpha - X_2^\alpha)$$

9/30/2016 PHY 711 Fall 2016 -- Lecture 13 7

Coupled linear equations:

$$-\omega_\alpha^2 m_1 X_1^\alpha = k(X_2^\alpha - X_1^\alpha)$$

$$-\omega_\alpha^2 m_2 X_2^\alpha = k(X_1^\alpha - 2X_2^\alpha + X_3^\alpha)$$

$$-\omega_\alpha^2 m_3 X_3^\alpha = -k(X_3^\alpha - X_2^\alpha)$$

Matrix form:

$$\begin{pmatrix} k - \omega_\alpha^2 m_1 & -k & 0 \\ -k & 2k - \omega_\alpha^2 m_2 & -k \\ 0 & -k & k - \omega_\alpha^2 m_3 \end{pmatrix} \begin{pmatrix} X_1^\alpha \\ X_2^\alpha \\ X_3^\alpha \end{pmatrix} = 0$$

9/30/2016 PHY 711 Fall 2016 -- Lecture 13 8

Matrix form:

$$\begin{pmatrix} k - \omega_\alpha^2 m_1 & -k & 0 \\ -k & 2k - \omega_\alpha^2 m_2 & -k \\ 0 & -k & k - \omega_\alpha^2 m_3 \end{pmatrix} \begin{pmatrix} X_1^\alpha \\ X_2^\alpha \\ X_3^\alpha \end{pmatrix} = 0$$

More convenient form:

Let $Y_i \equiv \sqrt{m_i} X_i$ Equations for Y_i take the form:

$$\begin{pmatrix} \kappa_{11} - \omega_\alpha^2 & -\kappa_{12} & 0 \\ -\kappa_{12} & 2\kappa_{22} - \omega_\alpha^2 & -\kappa_{23} \\ 0 & -\kappa_{23} & \kappa_{33} - \omega_\alpha^2 \end{pmatrix} \begin{pmatrix} Y_1^\alpha \\ Y_2^\alpha \\ Y_3^\alpha \end{pmatrix} = 0$$

where $\kappa_{ij} = \kappa_{ji} \equiv \frac{k}{\sqrt{m_i m_j}}$

9/30/2016 PHY 711 Fall 2016 -- Lecture 13 9

Digression:

Eigenvalue properties of matrices $\mathbf{M}\mathbf{y}_\alpha = \lambda_\alpha \mathbf{y}_\alpha$

Hermitian matrix : $H_{ij} = H_{ji}^*$

Theorem for Hermitian matrices :

λ_α have real values and $\mathbf{y}_\alpha^H \cdot \mathbf{y}_\beta = \delta_{\alpha\beta}$

Unitary matrix : $\mathbf{U}\mathbf{U}^H = \mathbf{I}$

$|\lambda_\alpha| = 1$ and $\mathbf{y}_\alpha^H \cdot \mathbf{y}_\beta = \delta_{\alpha\beta}$

9/30/2016

PHY 711 Fall 2016 -- Lecture 13

10

Digression on matrices -- continued

Eigenvalues of a matrix are "invariant" under a similarity transformation

Eigenvalue properties of matrix: $\mathbf{M}\mathbf{y}_\alpha = \lambda_\alpha \mathbf{y}_\alpha$

Transformed matrix: $\mathbf{M}'\mathbf{y}'_\alpha = \lambda'_\alpha \mathbf{y}'_\alpha$

If $\mathbf{M}' = \mathbf{S}\mathbf{M}\mathbf{S}^{-1}$ then $\lambda'_\alpha = \lambda_\alpha$ and $\mathbf{S}^{-1}\mathbf{y}'_\alpha = \mathbf{y}_\alpha$

Proof $\mathbf{S}\mathbf{M}\mathbf{S}^{-1}\mathbf{y}'_\alpha = \lambda'_\alpha \mathbf{y}'_\alpha$

$\mathbf{M}(\mathbf{S}^{-1}\mathbf{y}'_\alpha) = \lambda'_\alpha (\mathbf{S}^{-1}\mathbf{y}'_\alpha)$

9/30/2016

PHY 711 Fall 2016 -- Lecture 13

11

Example of transformation:

Original problem written in eigenvalue form:

$$\begin{pmatrix} k/m_1 & -k/m_1 & 0 \\ -k/m_2 & 2k/m_2 & -k/m_2 \\ 0 & -k/m_3 & k/m_3 \end{pmatrix} \begin{pmatrix} X_1^\alpha \\ X_2^\alpha \\ X_3^\alpha \end{pmatrix} = \omega_\alpha^2 \begin{pmatrix} X_1^\alpha \\ X_2^\alpha \\ X_3^\alpha \end{pmatrix}$$

$$\text{Let } \mathbf{S} = \begin{pmatrix} \sqrt{m_1} & 0 & 0 \\ 0 & \sqrt{m_2} & 0 \\ 0 & 0 & \sqrt{m_3} \end{pmatrix}; \quad \mathbf{S}\mathbf{M}\mathbf{S}^{-1} = \begin{pmatrix} \kappa_{11} & -\kappa_{12} & 0 \\ -\kappa_{12} & 2\kappa_{22} & -\kappa_{23} \\ 0 & -\kappa_{23} & \kappa_{33} \end{pmatrix}$$

Let $\mathbf{Y} = \mathbf{S}\mathbf{X}$

$$\begin{pmatrix} \kappa_{11} & -\kappa_{12} & 0 \\ -\kappa_{12} & 2\kappa_{22} & -\kappa_{23} \\ 0 & -\kappa_{23} & \kappa_{33} \end{pmatrix} \begin{pmatrix} Y_1^\alpha \\ Y_2^\alpha \\ Y_3^\alpha \end{pmatrix} = \omega_\alpha^2 \begin{pmatrix} Y_1^\alpha \\ Y_2^\alpha \\ Y_3^\alpha \end{pmatrix}$$

where $\kappa_{ij} = \kappa_{ji} \equiv \frac{k}{\sqrt{m_i m_j}}$

9/30/2016

PHY 711 Fall 2016 -- Lecture 13

12

In our case :

$$\begin{pmatrix} \kappa_{11} & -\kappa_{12} & 0 \\ -\kappa_{12} & 2\kappa_{22} & -\kappa_{23} \\ 0 & -\kappa_{23} & \kappa_{33} \end{pmatrix} \begin{pmatrix} Y_1^\alpha \\ Y_2^\alpha \\ Y_3^\alpha \end{pmatrix} = \omega_\alpha^2 \begin{pmatrix} Y_1^\alpha \\ Y_2^\alpha \\ Y_3^\alpha \end{pmatrix}$$

for $m_1 = m_3 \equiv m_o$ and $m_2 \equiv m_c$ (CO₂)

$$\begin{pmatrix} \kappa_{oo} & -\kappa_{oc} & 0 \\ -\kappa_{oc} & 2\kappa_{cc} & -\kappa_{oc} \\ 0 & -\kappa_{oc} & \kappa_{oo} \end{pmatrix} \begin{pmatrix} Y_1^\alpha \\ Y_2^\alpha \\ Y_3^\alpha \end{pmatrix} = \omega_\alpha^2 \begin{pmatrix} Y_1^\alpha \\ Y_2^\alpha \\ Y_3^\alpha \end{pmatrix}$$

9/30/2016 PHY 711 Fall 2016 -- Lecture 13 13

Eigenvalues and eigenvectors :

$$\omega_1^2 = 0 \quad \begin{pmatrix} Y_1^1 \\ Y_2^1 \\ Y_3^1 \end{pmatrix} = N_1 \begin{pmatrix} \sqrt{\frac{m_o}{m_c}} \\ 1 \\ \sqrt{\frac{m_o}{m_c}} \end{pmatrix}, \quad \begin{pmatrix} X_1^1 \\ X_2^1 \\ X_3^1 \end{pmatrix} = N'_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\omega_2^2 = \frac{k}{m_o} \quad \begin{pmatrix} Y_1^2 \\ Y_2^2 \\ Y_3^2 \end{pmatrix} = N_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} X_1^2 \\ X_2^2 \\ X_3^2 \end{pmatrix} = N'_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\omega_3^2 = \frac{k}{m_o} + \frac{2k}{m_c} \quad \begin{pmatrix} Y_1^3 \\ Y_2^3 \\ Y_3^3 \end{pmatrix} = N_3 \begin{pmatrix} 1 \\ -2\sqrt{\frac{m_o}{m_c}} \\ 1 \end{pmatrix}, \quad \begin{pmatrix} X_1^3 \\ X_2^3 \\ X_3^3 \end{pmatrix} = N'_3 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

9/30/2016 PHY 711 Fall 2016 -- Lecture 13 14

$\omega_1 = 0$

$\omega_2 = \sqrt{\frac{k}{m_o}}$

$\omega_3 = \sqrt{\frac{k}{m_o} + \frac{2k}{m_c}}$

9/30/2016 PHY 711 Fall 2016 -- Lecture 13 15

General solution :

$$x_i(t) = \Re \left(\sum_{\alpha} C^{\alpha} X_i^{\alpha} e^{-i\omega_{\alpha} t} \right)$$

For example, normal mode amplitudes C^{α} can be determined from initial conditions

9/30/2016 PHY 711 Fall 2016 -- Lecture 13 16

Additional digression on matrix properties
Singular value decomposition

It is possible to factor any real matrix \mathbf{A} into unitary matrices \mathbf{V} and \mathbf{U} together with positive diagonal matrix $\mathbf{\Sigma}$:

$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^H$$

$$\mathbf{\Sigma} = \begin{pmatrix} \sigma_1 & 0 & \dots & 0 \\ 0 & \sigma_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_N \end{pmatrix}$$

9/30/2016 PHY 711 Fall 2016 -- Lecture 13 17

Singular value decomposition -- continued

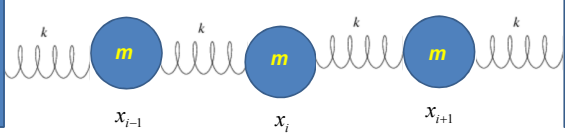
Consider using SVD to solve a singular linear algebra problem $\mathbf{A}\mathbf{X} = \mathbf{B}$

$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^H$$

$$\mathbf{X} = \sum_{i \text{ for } \sigma_i > \epsilon} \mathbf{v}_i \frac{\langle \mathbf{u}_i^H | \mathbf{B} \rangle}{\sigma_i}$$

9/30/2016 PHY 711 Fall 2016 -- Lecture 13 18

Consider an extended system of masses and springs:



Note: each mass coordinate is measured relative to its equilibrium position x_i^0

$$L = T - V = \frac{1}{2} m \sum_{i=1}^N \dot{x}_i^2 - \frac{1}{2} k \sum_{i=0}^N (x_{i+1} - x_i)^2$$

Note: In fact, we have N masses; x_0 and x_{N+1} will be treated using boundary conditions.

9/30/2016 PHY 711 Fall 2016 -- Lecture 13 19

$$L = T - V = \frac{1}{2} m \sum_{i=1}^N \dot{x}_i^2 - \frac{1}{2} k \sum_{i=0}^N (x_{i+1} - x_i)^2$$

$x_0 \equiv 0$ and $x_{N+1} \equiv 0$

From Euler - Lagrange equations :

$$m\ddot{x}_1 = k(x_2 - 2x_1)$$

$$m\ddot{x}_2 = k(x_3 - 2x_2 + x_1)$$

.....

$$m\ddot{x}_i = k(x_{i+1} - 2x_i + x_{i-1})$$

.....

$$m\ddot{x}_N = k(x_{N-1} - 2x_N)$$

9/30/2016 PHY 711 Fall 2016 -- Lecture 13 20

From Euler - Lagrange equations :

$$m\ddot{x}_j = k(x_{j+1} - 2x_j + x_{j-1}) \quad \text{with } x_0 = 0 = x_{N+1}$$

Try: $x_j(t) = A e^{-i\omega t + iqa_j}$

$$-\omega^2 A e^{-i\omega t + iqa_j} = \frac{k}{m} (e^{iqa} - 2 + e^{-iqa}) A e^{-i\omega t + iqa_j}$$

$$-\omega^2 = \frac{k}{m} (2 \cos(qa) - 2)$$

$$\Rightarrow \omega^2 = \frac{4k}{m} \sin^2\left(\frac{qa}{2}\right)$$

9/30/2016 PHY 711 Fall 2016 -- Lecture 13 21

From Euler-Lagrange equations -- continued:

$$m\ddot{x}_j = k(x_{j+1} - 2x_j + x_{j-1}) \quad \text{with } x_0 = 0 = x_{N+1}$$

Try: $x_j(t) = Ae^{-i\omega t + iqa_j} \Rightarrow \omega^2 = \frac{4k}{m} \sin^2\left(\frac{qa}{2}\right)$

Note that: $x_j(t) = Be^{-i\omega t - iqa_j} \Rightarrow \omega^2 = \frac{4k}{m} \sin^2\left(\frac{qa}{2}\right)$

General solution:

$$x_j(t) = \Re\left(Ae^{-i\omega t + iqa_j} + Be^{-i\omega t - iqa_j}\right)$$

Impose boundary conditions:

$$x_0(t) = \Re\left(Ae^{-i\omega t} + Be^{-i\omega t}\right) = 0$$

$$x_{N+1}(t) = \Re\left(Ae^{-i\omega t + iqa(N+1)} + Be^{-i\omega t - iqa(N+1)}\right) = 0$$

9/30/2016 PHY 711 Fall 2016 -- Lecture 13 22

Impose boundary conditions -- continued:

$$x_0(t) = \Re\left(Ae^{-i\omega t} + Be^{-i\omega t}\right) = 0$$

$$x_{N+1}(t) = \Re\left(Ae^{-i\omega t + iqa(N+1)} + Be^{-i\omega t - iqa(N+1)}\right) = 0$$

$\Rightarrow B = -A$

$$x_{N+1}(t) = \Re\left(Ae^{-i\omega t}\left(e^{iqa(N+1)} - e^{-iqa(N+1)}\right)\right) = 0$$

$\Rightarrow \sin(qa(N+1)) = 0$

$\Rightarrow qa(N+1) = \nu\pi \quad \text{where } \nu = 0, 1, 2, \dots$

$$qa = \frac{\nu\pi}{N+1}$$

9/30/2016 PHY 711 Fall 2016 -- Lecture 13 23

Summary of results:

$$\Rightarrow \omega_\nu^2 = \frac{4k}{m} \sin^2\left(\frac{\nu\pi}{2(N+1)}\right) \quad x_n = \Re\left(2iA \sin\left(\frac{\nu\pi n}{N+1}\right)\right)$$

$\nu = 0, 1, \dots, N \quad n = 1, 2, \dots, N$

9/30/2016 PHY 711 Fall 2016 -- Lecture 13 24
