PHY 711 Classical Mechanics and Mathematical Methods
11-11:50 AM MWF Olin 107

Plan for Lecture 17:

Read Chapter 7 & Appendices A-D

Generalization of the one dimensional wave equation → various mathematical problems and techniques including:

1. **Sturm-Liouville equations**
2. **Orthogonal function expansions**
3. **Green's functions methods**

	Mon, 9/05/2016		Labor day -- no class		
3	Wed, 9/07/2016	Chap. 1	Scattering theory	#3	9/9/2016
4	Fri, 9/09/2016	Chap. 1 & 2	Scattering theory and rotations	#4	9/12/2016
5	Mon, 9/12/2016	Chap. 3	Calculus of variations	#5	9/14/2016
6	Wed, 9/14/2016	Chap. 3	Calculus of variations	#6	9/16/2016
7	Fri, 9/16/2016	Chap. 3	Lagrangian mechanics	#7	9/19/2016
8	Mon, 9/19/2016	Chap. 3 and 6	Lagrangian mechanics and constraints	#8	9/21/2016
9	Wed, 9/21/2016	Chap. 3 and 6	Constants of the motion	#9	9/23/2016
10	Fri, 9/23/2016	Chap. 3 and 6	Hamiltonian and canonical equations of motion	#10	9/26/2016
11	Mon, 9/26/2016	Chap. 3 and 6	Phase space	#11	9/28/2016
12	Wed, 9/28/2016	Chap. 6	Canonical transformations	#12	9/30/2016
13	Fri, 9/30/2016	Chap. 4	Small oscillations	#13	10/04/2016
14	Tue, 10/04/2016	Chap. 4	Normal modes	#14	10/07/2016
15	Wed, 10/05/2016	Chap. 7	Wave motion in one dimension	#15	10/07/2016
16	Fri, 10/07/2016	Chap. 7	Sturm-Liouville equations		
17	Mon, 10/10/2016	Chap. 7	Sturm-Liouville equations	Take-home exam	
18	Wed, 10/12/2016			Take-home exam	
19	Fri, 10/14/2016			Take-home exam	
20	Mon, 10/17/2016			Exam due	
21	Wed, 10/19/2016				
	Fri, 10/21/2016		Fall break -- no class		
22	Mon, 10/24/2016				

Eigenvalues and eigenfunctions of Sturm-Liouville equations

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)\right)f_n(x)=\lambda_n\sigma(x)f_n(x)$$

Properties:

Eigenvalues λ_n are real

Eigenfunctions are orthogonal: $\int_a^b \sigma(x)f_n(x)f_m(x)dx=\delta_{nm}N_n,$

where $N_n \equiv \int_a^b \sigma(x)(f_n(x))^2dx.$

Variation approximation to lowest eigenvalue

In general, there are several techniques to determine the eigenvalues λ_n and eigenfunctions $f_n(x)$. When it is not possible to find the ``exact'' functions, there are several powerful approximation techniques. For example, the lowest eigenvalue can be approximated by minimizing the function

$$\lambda_0 \le \frac{\langle \tilde{h}|S|\tilde{h}\rangle}{\langle \tilde{h}|\sigma|\tilde{h}\rangle}, \qquad S(x) \equiv -\frac{d}{dx}\tau(x)\frac{d}{dx} + v(x)$$

where $\tilde{h}(x)$ is a variable function which satisfies the correct boundary values. The ``proof'' of this inequality is based on the notion that $\tilde{h}(x)$ can in principle be expanded in terms of the (unknown) exact eigenfunctions $f_n(x)$: $\tilde{h}(x) = \sum_n C_n f_n(x)$, where the coefficients C_n can be assumed to be real.

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Estimation of the lowest eigenvalue – continued:

From the eigenfunction equation, we know that

$$S(x)\tilde{h}(x) = S(x)\sum_n C_n f_n(x) = \sum_n C_n \lambda_n \sigma(x) f_n(x).$$

It follows that:

$$\langle \tilde{h}|S|\tilde{h}\rangle = \int_a^b \tilde{h}(x)S(x)\tilde{h}(x)dx = \sum_n |C_n|^2 N_n \lambda_n.$$

It also follows that:

$$\langle \tilde{h}|\sigma|\tilde{h}\rangle = \int_a^b \tilde{h}(x)\sigma(x)\tilde{h}(x)dx = \sum_n |C_n|^2 N_n,$$

$$\text{Therefore} \quad \frac{\langle \tilde{h}|S|\tilde{h}\rangle}{\langle \tilde{h}|\sigma|\tilde{h}\rangle} = \frac{\sum_n |C_n|^2 N_n \lambda_n}{\sum_n |C_n|^2 N_n} \ge \lambda_0.$$

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Rayleigh-Ritz method of estimating the lowest eigenvalue

$$\lambda_0 \le \frac{\langle \tilde{h}|S|\tilde{h}\rangle}{\langle \tilde{h}|\sigma|\tilde{h}\rangle},$$

$$\text{Example:} \quad -\frac{d^2}{dx^2} f_n(x) = \lambda_n f_n(x) \qquad \text{with } f_n(0) = f_n(a) = 0$$

$$\text{trial function} \quad f_{\text{trial}}(x) = x(x-a)$$

$$\text{Exact value of } \lambda_0 = \frac{\pi^2}{a^2} = \frac{9.869604404}{a^2}$$

$$\text{Raleigh-Ritz estimate:} \quad \frac{\langle x(a-x)| -\frac{d^2}{dx^2} |x(a-x)\rangle}{\langle x(a-x)|x(a-x)\rangle} = \frac{10}{a^2}$$

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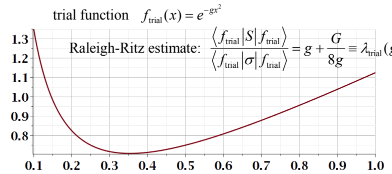
Rayleigh-Ritz method of estimating the lowest eigenvalue

$$\lambda_0 \le \frac{\langle \tilde{h} | S | \tilde{h} \rangle}{\langle \tilde{h} | \sigma | \tilde{h} \rangle},$$

Example: $-\frac{d^2 f_n(x)}{dx^2} + Gx^2 f_n(x) = \lambda_n f_n(x)$ with $f_n(-\infty) = f_n(\infty) = 0$

trial function $f_{\text{trial}}(x) = e^{-gx^2}$

Raleigh-Ritz estimate: $\frac{\langle f_{\text{trial}} | S | f_{\text{trial}} \rangle}{\langle f_{\text{trial}} | \sigma | f_{\text{trial}} \rangle} = g + \frac{G}{8g} \equiv \lambda_{\text{trial}}(g)$



$$g_0 = \sqrt{\frac{G}{8}} \qquad \lambda_{\text{trial}}(g_0) = \sqrt{\frac{G}{2}}$$

Comment on “completeness” of set of eigenfunctions

It can be shown that for any reasonable function *h(x)*, defined within the interval *a < x <b*, we can expand that function as a linear combination of the eigenfunctions $f_n(x)$

$$h(x) \approx \sum_n C_n f_n(x),$$

where $C_n = \frac{1}{N_n}\int_a^b \sigma(x')h(x')f_n(x')dx'.$

These ideas lead to the notion that the set of eigenfunctions $f_n(x)$ form a ``complete'' set in the sense of ``spanning'' the space of all functions in the interval *a < x <b,* as summarized by the statement:

$$\sigma(x)\sum_n \frac{f_n(x) f_n(x')}{N_n} = \delta(x-x').$$

Green’s function solution methods

Suppose that we can find a Green's function defined as follows:

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx} + v(x) - \lambda\sigma(x)\right)G_\lambda(x,x') = \delta(x-x')$$

Completeness of eigenfunctions:

Recall:

$$\sigma(x)\sum_n \frac{f_n(x) f_n(x')}{N_n} = \delta(x-x')$$

In terms of eigenfunctions:

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx} + v(x) - \lambda\sigma(x)\right)G_\lambda(x,x') = \sigma(x)\sum_n \frac{f_n(x) f_n(x')}{N_n}$$

$$\Rightarrow G_\lambda(x,x') = \sum_n \frac{f_n(x) f_n(x')/N_n}{\lambda_n - \lambda}$$

Solution to inhomogeneous problem by using Green's functions

Inhomogenous problem:

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)\varphi(x)=F(x)$$

Green's function :

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)G_{\lambda}(x,x')=\delta(x-x')$$

Formal solution:

$$\varphi_{\lambda}(x)=\varphi_{\lambda 0}(x)+\int_0^L G_{\lambda}(x,x')F(x')dx'$$

Solution to homogeneous problem

Example Sturm-Liouville problem:

Example: $\tau(x)=1;\quad \sigma(x)=1;\quad v(x)=0;\quad a=0$ and $b=L$

$\lambda=1;\quad F(x)=F_0\sin\left(\frac{\pi x}{L}\right)$

Inhomogenous equation :

$$\left(-\frac{d^2}{dx^2}-1\right)\phi(x)=F_0\sin\left(\frac{\pi x}{L}\right)$$

Eigenvalue equation :

$$\left(-\frac{d^2}{dx^2}\right)f_n(x)=\lambda_n f_n(x)$$

Eigenfunctions

$$f_n(x)=\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)$$

Eigenvalues :

$$\lambda_n=\left(\frac{n\pi}{L}\right)^2$$

Completeness of eigenfunctions :

$$\sigma(x)\sum_n\frac{f_n(x)f_n(x')}{N_n}=\delta(x-x')$$

In this example : $\frac{2}{L}\sum_n\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi x'}{L}\right)=\delta(x-x')$

Green's function :

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)G_\lambda(x,x')=\delta(x-x')$$

Green's function for the example :

$$G(x,x')=\sum_n \frac{f_n(x)f_n(x')/N_n}{\lambda_n-\lambda}=\frac{2}{L}\sum_n \frac{\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi x'}{L}\right)}{\left(\frac{n\pi}{L}\right)^2-1}$$

Using Green's function to solve inhomogenous equation :

$$\left(-\frac{d^2}{dx^2}-1\right)\phi(x)=F_0\sin\left(\frac{\pi x}{L}\right)$$

$$\phi(x)=\phi_0(x)+\int_0^L G(x,x')F_0\sin\left(\frac{\pi x'}{L}\right)dx'$$

$$=\phi_0(x)+\frac{2}{L}\sum_n\left[\frac{\sin\left(\frac{n\pi x}{L}\right)}{\left(\frac{n\pi}{L}\right)^2-1}\int_0^L \sin\left(\frac{n\pi x'}{L}\right)F_0\sin\left(\frac{\pi x'}{L}\right)dx'\right]$$

$$=\phi_0(x)+\frac{F_0}{\left(\frac{\pi}{L}\right)^2-1}\sin\left(\frac{\pi x}{L}\right)$$

Alternate Green's function method:

$$G(x,x')=\frac{1}{W}g_a(x_<)g_b(x_>)$$

$$\left(-\frac{d^2}{dx^2}-1\right)g_i(x)=0 \qquad \Rightarrow g_a(x)=\sin(x); \qquad g_b(x)=\sin(L-x);$$

$$W=g_b(x)\frac{dg_a(x)}{dx}-g_a(x)\frac{dg_b(x)}{dx}=\sin(L-x)\cos(x)+\sin(x)\cos(L-x)$$

$$=\sin(L)$$

$$\varphi(x)=\varphi_0(x)+\frac{\sin(L-x)}{\sin(L)}\int_0^x \sin(x')F_0\sin\left(\frac{\pi x'}{L}\right)dx'$$

$$+\frac{\sin(x)}{\sin(L)}\int_x^L \sin(L-x')F_0\sin\left(\frac{\pi x'}{L}\right)dx'$$

$$\varphi(x)=\varphi_0(x)+\frac{F_0}{\left(\frac{\pi}{L}\right)^2-1}\sin\left(\frac{\pi x}{L}\right)$$

(after some algebra)

General method of constructing Green's functions using homogeneous solution

Green's function :

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)G_\lambda(x,x')=\delta(x-x')$$

Two homogeneous solutions

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)g_i(x)=0 \qquad \text{for} \quad i=a,b$$

Let

$$G_\lambda(x,x')=\frac{1}{W}g_a(x_<)g_b(x_>)$$

For $\epsilon \to 0$:

$$\int_{x'-\epsilon}^{x'+\epsilon} dx\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)G_\lambda(x,x')=\int_{x'-\epsilon}^{x'+\epsilon} dx\,\delta(x-x')$$

$$\int_{x'-\epsilon}^{x'+\epsilon} dx\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}\right)\frac{1}{W}g_a(x_<)g_b(x_>)=1$$

$$-\frac{\tau(x)}{W}\left(\frac{d}{dx}g_a(x_<)g_b(x_>)\right)\Bigg]_{x'-\epsilon}^{x'+\epsilon}=\frac{\tau(x')}{W}\left(g_a(x')\frac{d}{dx}g_b(x')-g_b(x')\frac{d}{dx}g_a(x')\right)$$

$$\Rightarrow W=\tau(x')\left(g_a(x')\frac{d}{dx}g_b(x')-g_b(x')\frac{d}{dx}g_a(x')\right)$$

Note -- W (Wronskian) is constant, since $\frac{dW}{dx'}=0.$

$\Rightarrow$ Useful Green's function construction in one dimension:

$$G_\lambda(x,x')=\frac{1}{W}g_a(x_<)g_b(x_>)$$

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)\varphi(x)=F(x)$$

Green's function solution:

$$\varphi_\lambda(x)=\varphi_{\lambda 0}(x)+\int_{x_l}^{x_u} G_\lambda(x,x')F(x')dx'$$

$$=\varphi_{\lambda 0}(x)+\frac{g_b(x)}{W}\int_{x_l}^{x} g_a(x')F(x')dx'+\frac{g_a(x)}{W}\int_{x}^{x_u} g_b(x')F(x')dx'$$