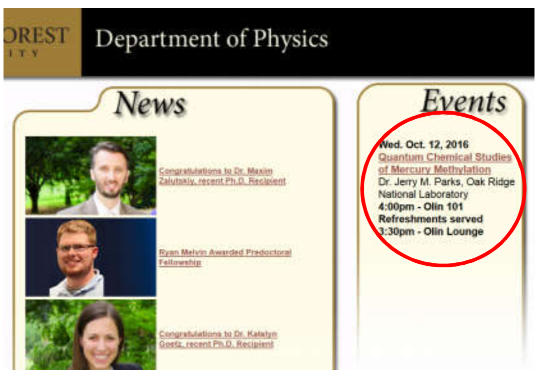
PHY 711 Classical Mechanics and Mathematical Methods
11-11:50 AM MWF Olin 107

Plan for Lecture 18:

Read Chapter 7 & Appendices A-D

Generalization of the one dimensional wave equation → various mathematical problems and techniques including:

1. **Orthogonal function expansions**
2. **Fourier series**
3. **Fourier transforms**
4. **Fast Fourier transforms**



3	Wed, 9/07/2016	Chap. 1	Scattering theory	#3	9/9/2016
4	Fri, 9/09/2016	Chap. 1 & 2	Scattering theory and rotations	#4	9/12/2016
5	Mon, 9/12/2016	Chap. 3	Calculus of variations	#5	9/14/2016
6	Wed, 9/14/2016	Chap. 3	Calculus of variations	#6	9/16/2016
7	Fri, 9/16/2016	Chap. 3	Lagrangian mechanics	#7	9/19/2016
8	Mon, 9/19/2016	Chap. 3 and 6	Lagrangian mechanics and constraints	#8	9/21/2016
9	Wed, 9/21/2016	Chap. 3 and 6	Constants of the motion	#9	9/23/2016
10	Fri, 9/23/2016	Chap. 3 and 6	Hamiltonian and canonical equations of motion	#10	9/26/2016
11	Mon, 9/26/2016	Chap. 3 and 6	Phase space	#11	9/28/2016
12	Wed, 9/28/2016	Chap. 6	Canonical transformations	#12	9/30/2016
13	Fri, 9/30/2016	Chap. 4	Small oscillations	#13	10/04/2016
14	Tue, 10/04/2016	Chap. 4	Normal modes	#14	10/07/2016
15	Wed, 10/05/2016	Chap. 7	Wave motion in one dimension	#15	10/07/2016
16	Fri, 10/07/2016	Chap. 7	Sturm-Liouville equations		
17	Mon, 10/10/2016	Chap. 7	Sturm-Liouville equations	Take-home exam	
18	Wed, 10/12/2016	Chap. 7	Fourier series and transforms	Take-home exam	
19	Fri, 10/14/2016			Take-home exam	
20	Mon, 10/17/2016			Exam due	
21	Wed, 10/19/2016				
	Fri, 10/21/2016		Fall break -- no class		
22	Mon, 10/24/2016				

Eigenvalues and eigenfunctions of Sturm-Liouville equations

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)\right)f_n(x)=\lambda_n\sigma(x)f_n(x)$$

Properties:

Eigenvalues λ_n are real

Eigenfunctions are orthogonal: $\int_a^b \sigma(x)f_n(x)f_m(x)dx=\delta_{nm}N_n,$

where $N_n\equiv\int_a^b\sigma(x)(f_n(x))^2dx.$

Special case: $\tau(x)=1=\sigma(x)\qquad v(x)=0$

$$-\frac{d^2}{dx^2}f_n(x)=\lambda_n f_n(x)\quad\text{for }0\le x\le a,\quad\text{with } f_n(0)=f_n(a)=0$$

Special case: $\tau(x)=1=\sigma(x)\qquad v(x)=0$

$$-\frac{d^2}{dx^2}f_n(x)=\lambda_n f_n(x)\quad\text{for }0\le x\le a,\quad\text{with } f_n(0)=f_n(a)=0$$

$$f_n(x)=\sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)\qquad\qquad \lambda_n=\left(\frac{n\pi}{a}\right)^2$$

Fourier series representation of function $h(x)$ in the interval $0\le x\le a$:

$$h(x)=\sum_{n=1}^{\infty}A_n\sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)$$

$$A_n=\sqrt{\frac{2}{a}}\int_0^a dx'\ h(x')\ \sin\left(\frac{n\pi x'}{a}\right)$$

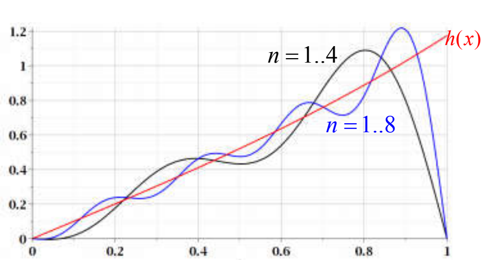
*Note that if $h(x)$ does not vanish at $x=0$ and $x=a$, the more general expression applies:

$$h(x)=\sum_{n=1}^{\infty}A_n\sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)+\sum_{n=0}^{\infty}B_n\sqrt{\frac{2}{a}}\cos\left(\frac{n\pi x}{a}\right)$$

(with some restrictions).

Example

$$h(x)=\sinh(x)\approx 2\pi\sinh(1)\left(\frac{\sin(\pi x)}{\pi^2+1}-\frac{2\sin(2\pi x)}{4\pi^2+1}+\ldots-(-1)^n n\frac{\sin(n\pi x)}{n^2\pi^2+1}+\ldots\right)$$



Fourier series representation of function $h(x)$ in the interval $0 \le x \le a$:

$$h(x) = \sum_{n=1}^{\infty} A_n \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)$$

$$A_n = \sqrt{\frac{2}{a}} \int_0^a dx' \ h(x') \ \sin\left(\frac{n\pi x'}{a}\right)$$

Can show that the series converges provided that $h(x)$ is piecewise continuous.

Generalization to infinite range

Examples in time domain --

10/12/2016 PHY 711 Fall 2016 -- Lecture 18 7

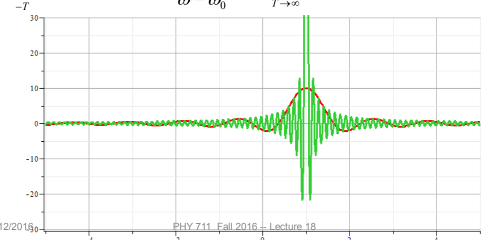
Fourier transforms

A useful identity

$$\int_{-\infty}^{\infty} dt \ e^{-i(\omega-\omega_0)t} = 2\pi\delta(\omega-\omega_0)$$

Note that

$$\int_{-T}^{T} dt \ e^{-i(\omega-\omega_0)t} = \frac{2\sin[(\omega-\omega_0)T]}{\omega-\omega_0} \underset{T\to\infty}{\approx} 2\pi\delta(\omega-\omega_0)$$



10/12/2016 PHY 711 Fall 2016 -- Lecture 18 8

Definition of Fourier Transform for a function $f(t)$:

$$f(t) = \int_{-\infty}^{\infty} d\omega \, F(\omega) \, e^{-i\omega t}$$

Backward transform:

$$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \, f(t) \, e^{i\omega t}$$

Check:

$$f(t) = \int_{-\infty}^{\infty} d\omega \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} dt' \ f(t') \ e^{i\omega t'} \right) e^{-i\omega t}$$

$$f(t) = \int_{-\infty}^{\infty} dt' \ f(t') \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \ e^{i\omega(t'-t)} \right) = \int_{-\infty}^{\infty} dt' \ f(t') \ \delta(t'-t)$$

Note: The location of the 2π factor varies among texts.

10/12/2016 PHY 711 Fall 2016 -- Lecture 18 9

Properties of Fourier transforms -- Parseval's theorem:

$$\int_{-\infty}^{\infty} dt\,\left(f(t)\right)^* f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\,\left(F(\omega)\right)^* F(\omega)$$

Check: $$\int_{-\infty}^{\infty} dt\,\left(f(t)\right)^* f(t) = \int_{-\infty}^{\infty} dt\left(\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega F(\omega)e^{i\omega t}\right)^* \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega' F(\omega')e^{i\omega' t}\right)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega F^*(\omega)\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega' F(\omega')\int_{-\infty}^{\infty} dt\; e^{i(\omega'-\omega)t}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega F^*(\omega)\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega' F(\omega')\; 2\pi\delta(\omega'-\omega)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega F^*(\omega)F(\omega)$$

Use of Fourier transforms to solve wave equation

Wave equation : $\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0$

Suppose $u(x,t) = e^{-i\omega t}\tilde{F}(x,\omega)$ where $\tilde{F}(x,\omega)$ satisfies the equation :

$$\frac{\partial^2 \tilde{F}(x,\omega)}{\partial x^2} = -\frac{\omega^2}{c^2}\tilde{F}(x,\omega) \equiv -k^2\tilde{F}(x,\omega)$$

More generally:

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega F(x,\omega) e^{-i\omega t}$$

Further assume that fixed boundary conditions apply : $0 \le x \le L$

with $\tilde{F}(0,\omega) = 0$ and $\tilde{F}(L,\omega) = 0$

For $n = 1,2,3\cdots$

$$\tilde{F}_n(x,\omega) = \sin\left(\frac{n\pi x}{L}\right) \qquad k \to k_n = \frac{n\pi}{L} \equiv \frac{\omega_n}{c}$$

$$u(x,t) = e^{-i\omega_n t}\sin(k_n x) = e^{-i\omega_n t}\frac{\left(e^{ik_n x} - e^{-ik_n x}\right)}{2i} = \frac{\left(e^{ik_n(x-ct)} - e^{-ik_n(x+ct)}\right)}{2i}$$

Use of Fourier transforms to solve wave equation -- continued

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0$$

Using superposition : Suppose $u(x,t) = \sum_n C_n e^{-i\omega_n t}\tilde{F}_n(x,\omega_n)$

$$\frac{\partial^2 \tilde{F}_n(x,\omega_n)}{\partial x^2} = -\frac{\omega_n^2}{c^2}\tilde{F}(x,\omega_n) \equiv -k_n^2\tilde{F}(x,\omega_n)$$

For $\tilde{F}_n(x,\omega) = \sin\left(\frac{n\pi x}{L}\right) \qquad k \to k_n = \frac{n\pi}{L} \equiv \frac{\omega_n}{c}$

$$\Rightarrow u(x,t) = \sum_n C_n e^{-i\omega_n t}\sin(k_n x) = \sum_n \frac{C_n}{2i}e^{-i\omega_n t}\left(e^{ik_n x} - e^{-ik_n x}\right)$$

$$= \sum_n \frac{C_n}{2i}\left(e^{ik_n(x-ct)} - e^{-ik_n(x+ct)}\right) \equiv f(x-ct) + g(x+ct)$$

Fourier transform for a time periodic function:

Suppose $f(t+nT)=f(t)$ for and integer n

$$F(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty} dt\; f(t)e^{i\omega t} \;=\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\left(\int_0^T dt\; f(t)e^{i\omega(t+nT)}\right)$$

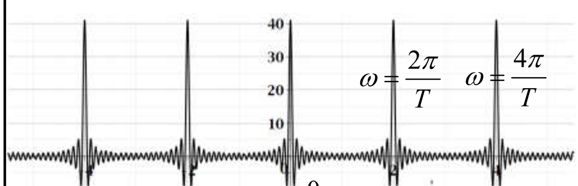
Note that:

$$\sum_{n=-\infty}^{\infty} e^{in\omega T}=\Omega\sum_{\nu=-\infty}^{\infty}\delta(\omega-\nu\Omega),\quad \text{where}\quad \Omega\equiv\frac{2\pi}{T}$$

Details:

$$\sum_{n=-\infty}^{\infty} e^{in\omega T} = \lim_{N\to\infty}\sum_{n=-N}^{N} e^{in\omega T} = \lim_{N\to\infty}\frac{\sin\left(\left(N+\frac{1}{2}\right)\omega \mathrm{T}\right)}{\sin\left(\frac{1}{2}\omega \mathrm{T}\right)}$$

$$\frac{\sin\left(\left(N+\frac{1}{2}\right)\omega \mathrm{T}\right)}{\sin\left(\frac{1}{2}\omega \mathrm{T}\right)}$$



Note that:

$$\sum_{n=-\infty}^{\infty} e^{in\omega T}=\Omega\sum_{\nu=-\infty}^{\infty}\delta(\omega-\nu\Omega),\quad \text{where}\quad \Omega\equiv\frac{2\pi}{T}$$

Some details :

$$\sum_{n=-M}^{M} e^{in\omega T} = \frac{\sin\left(\left(M+\frac{1}{2}\right)\omega T\right)}{\sin\left(\frac{1}{2}\omega T\right)}$$

$$\lim_{M\to\infty}\left(\frac{\sin\left(\left(M+\frac{1}{2}\right)\omega T\right)}{\sin\left(\frac{1}{2}\omega T\right)}\right)=2\pi\sum_{\nu}\delta(\omega T-\nu\Omega T)=\frac{2\pi}{T}\sum_{\nu}\delta(\omega-\nu\Omega)$$

$$\Rightarrow \sum_{n=-\infty}^{\infty} e^{in\omega T} =\Omega\sum_{\nu=-\infty}^{\infty}\delta(\omega-\nu\Omega),\quad \text{where}\quad \Omega\equiv\frac{2\pi}{T}$$

$$\Rightarrow F(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, f(t)e^{i\omega t} \;=\sum_{\nu=-\infty}^{\infty}\Omega\delta(\omega-\nu\Omega)\left(\int_0^T dt\, f(t)e^{i\omega t}\right)$$

Thus, for a time periodic function

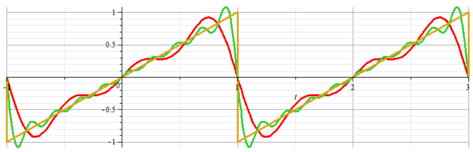
$$f(t)=\frac{1}{2\pi}\sum_{\nu=-\infty}^{\infty}F(\nu\Omega)e^{-i\nu\Omega t}$$

Example:

Suppose: $f(t)=\begin{cases}\dfrac{t-nT}{T} & \text{for } (n-1)T\le t\le(n+1)T;\ \ n=0,2,4,6\ldots\\ 0 & \text{otherwise}\end{cases}$

Note, in this case the repeat period is $2T$ and the convenient sample time interval is $-T\le t\le T$.

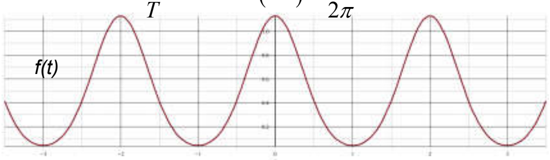
$$\bar{F}(\nu\Omega)=\frac{1}{2T}i\int_{-T}^{T}\frac{t}{T}\sin\left(\frac{\nu 2\pi t}{2T}\right)dt \qquad f(t)=\sum_{\nu=1}^{\infty}2\left|\bar{F}(\nu\Omega)\right|\sin\left(\frac{\nu 2\pi t}{2T}\right)$$

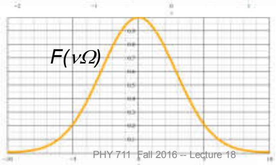


Example:

Suppose: $f(t)=\dfrac{1}{a\sqrt{\pi}}\displaystyle\sum_{n=-\infty}^{\infty}e^{-(t+nT)^2/a^2}=\sum_{\nu=-\infty}^{\infty}F(\nu\Omega)e^{-i\nu\Omega t}$

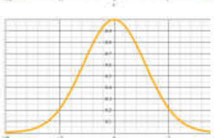
where $\Omega\equiv\dfrac{2\pi}{T}$ and $F(\nu\Omega)=\dfrac{1}{2\pi}e^{-a^2\nu^2\Omega^2/4}$

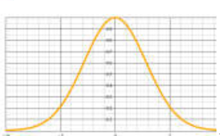


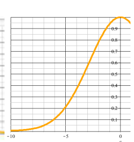


Continued: $f(t)=\dfrac{1}{a\sqrt{\pi}}\displaystyle\sum_{n=-\infty}^{\infty}e^{-(t+nT)^2/a^2}=\sum_{\nu=-\infty}^{\infty}F(\nu\Omega)e^{-i\nu\Omega t}$

Note: $f(t)\approx=\displaystyle\sum_{\nu=-M}^{M}F(\nu\Omega)e^{-i\nu\Omega t}$







ν=-M ν=M

$$\Rightarrow f\left(\frac{mT}{2M+1}\right)=\sum_{\nu=-M}^{M}F(\nu\Omega)e^{-i2\pi\nu m/(2M+1)}$$

Thus, for a periodic function

$$f(t)=\sum_{\nu=-\infty}^{\infty}F(\nu\Omega)e^{-i\nu\Omega t}$$

Now suppose that the transformed function is bounded;

$|F(\nu\Omega)|\leq\varepsilon$ for $|\nu|\geq N$

Define a periodic transform function function

$$\widetilde{F}(\nu\Omega)\equiv\widetilde{F}(\nu\Omega+\nu'((2N+1)\Omega))$$

Effect on time domain :

$$f(t)=\sum_{\nu=-\infty}^{\infty}\widetilde{F}(\nu\Omega)e^{-i\nu\Omega t}=\frac{2\pi}{(2N+1)\Omega}\sum_{\nu=-N}^{N}\widetilde{F}(\nu\Omega)e^{-i\nu\Omega t}\sum_{\mu}\delta\left(t-\frac{\mu T}{2N+1}\right)$$

10/12/2016 PHY 711 Fall 2016 -- Lecture 18 19

Doubly periodic functions

$$t\rightarrow\frac{\mu T}{2N+1}$$

$$\widetilde{f}_{\mu}=\frac{1}{2N+1}\sum_{\nu=-N}^{N}\widetilde{F}_{\nu}e^{-i2\pi\nu\mu/(2N+1)}$$

$$\widetilde{F}_{\nu}=\sum_{\mu=-N}^{N}\widetilde{f}_{\mu}e^{i2\pi\nu\mu/(2N+1)}$$

10/12/2016 PHY 711 Fall 2016 -- Lecture 18 20

More convenient notation

$$2N+1\rightarrow M$$

$$\widetilde{f}_{\mu}=\frac{1}{M}\sum_{\nu=0}^{M-1}\widetilde{F}_{\nu}e^{-i2\pi\nu\mu/M}$$

$$\widetilde{F}_{\nu}=\sum_{\mu=0}^{M}\widetilde{f}_{\mu}e^{i2\pi\nu\mu/M}$$

Note that for $W=e^{i2\pi/M}$

$$\widetilde{F}_0=\widetilde{f}_0W^0+\widetilde{f}_1W^0+\widetilde{f}_2W^0+\widetilde{f}_3W^0+\cdots$$

$$\widetilde{F}_1=\widetilde{f}_0W^0+\widetilde{f}_1W^1+\widetilde{f}_2W^2+\widetilde{f}_3W^3+\cdots$$

$$\widetilde{F}_2=\widetilde{f}_0W^0+\widetilde{f}_1W^2+\widetilde{f}_2W^4+\widetilde{f}_3W^6+\cdots$$

10/12/2016 PHY 711 Fall 2016 -- Lecture 18 21

Note that for $W = e^{i2\pi/M}$

$$\tilde{F}_0 = \tilde{f}_0 W^0 + \tilde{f}_1 W^0 + \tilde{f}_2 W^0 + \tilde{f}_3 W^0 + \cdots$$
$$\tilde{F}_1 = \tilde{f}_0 W^0 + \tilde{f}_1 W^1 + \tilde{f}_2 W^2 + \tilde{f}_3 W^3 + \cdots$$
$$\tilde{F}_2 = \tilde{f}_0 W^0 + \tilde{f}_1 W^2 + \tilde{f}_2 W^4 + \tilde{f}_3 W^6 + \cdots$$

However, $W^M = \left(e^{i2\pi/M}\right)^M = 1$

and $W^{M/2} = \left(e^{i2\pi/M}\right)^{M/2} = -1$

Cooley-Tukey algorithm: J. W. Cooley and J. W. Tukey, "An algorithm for machine calculation of complex Fourier series" Math. Computation 19, 297-301 (1965)