PHY 711 Classical Mechanics and Mathematical Methods
11-11:50 AM MWF Olin 107

Plan for Lecture 19:

Read Chapter 7 & Appendices A-D

Generalization of the one dimensional wave equation ➔ various mathematical problems and techniques including:

1. **Laplace transforms**
2. **Complex variables**
3. **Contour integrals**

4	Fri, 9/09/2016	Chap. 1 & 2	Scattering theory and rotations	#4	9/12/2016
5	Mon, 9/12/2016	Chap. 3	Calculus of variations	#5	9/14/2016
6	Wed, 9/14/2016	Chap. 3	Calculus of variations	#6	9/16/2016
7	Fri, 9/16/2016	Chap. 3	Lagrangian mechanics	#7	9/19/2016
8	Mon, 9/19/2016	Chap. 3 and 6	Lagrangian mechanics and constraints	#8	9/21/2016
9	Wed, 9/21/2016	Chap. 3 and 6	Constants of the motion	#9	9/23/2016
10	Fri, 9/23/2016	Chap. 3 and 6	Hamiltonian and canonical equations of motion	#10	9/26/2016
11	Mon, 9/26/2016	Chap. 3 and 6	Phase space	#11	9/28/2016
12	Wed, 9/28/2016	Chap. 6	Canonical transformations	#12	9/30/2016
13	Fri, 9/30/2016	Chap. 4	Small oscillations	#13	10/04/2016
14	Tue, 10/04/2016	Chap. 4	Normal modes	#14	10/07/2016
15	Wed, 10/05/2016	Chap. 7	Wave motion in one dimension	#15	10/07/2016
16	Fri, 10/07/2016	Chap. 7	Sturm-Liouville equations		
17	Mon, 10/10/2016	Chap. 7	Sturm-Liouville equations	Take-home exam	
18	Wed, 10/12/2016	Chap. 7	Fourier series and transforms	Take-home exam	
19	Fri, 10/14/2016	App. A	Laplace transforms and contour integrals	Take-home exam	
20	Mon, 10/17/2016			Exam due	
21	Wed, 10/19/2016				
	Fri, 10/21/2016		Fall break -- no class		
22	Mon, 10/24/2016				
23	Wed, 11/02/2016				

Laplace transforms

Laplace transforms can be used to solve initial value problems. The Laplace transform of a function $\phi(x)$ is defined as

$$\mathcal{L}_\phi(p) \equiv \int_0^\infty e^{-px}\phi(x)dx. \quad (24)$$

Assuming that $\phi(x)$ is well-behaved in the interval $0 \le x \le \infty$, the following properties are useful:

$$\mathcal{L}_{d\phi/dx}(p) = -\phi(0) + p\mathcal{L}_\phi(p), \quad (25)$$

and

$$\mathcal{L}_{d^2\phi/dx^2}(p) = -\frac{d\phi(0)}{dx} - p\phi(0) + p^2\mathcal{L}_\phi(p). \quad (26)$$

These identities allow us to turn a differential equation for $\phi(x)$ into an algebraic equation for $\mathcal{L}_\phi(p)$. We then need to perform an inverse Laplace transform to find $\phi(x)$.
For illustration, we will consider a simple example with $\tau(x) = 1$, $\sigma(x) = 1$, $\lambda = 0$. The differential equation then becomes

$$-\frac{d^2\phi(x)}{dx^2} = F(x), \tag{27}$$

where we will take the initial conditions to be $\phi(0) = 0$ and $d\phi(0)/dx = 0$. For our example, we will also take $F(x) = F_0 e^{-\gamma x}$. Multiplying, both sides of the equation by e^{-px} and integrating $0 \le x \le \infty$, we find

$$\mathcal{L}_\phi(p) = -\frac{F_0}{p^2(\gamma + p)}. \tag{28}$$

In general the inverse Laplace transform involves performing a contour integral, but we can use the following simple relations

$$\mathcal{L}_1 = \int_0^\infty e^{-px} dx = \frac{1}{p}. \tag{29}$$

$$\mathcal{L}_x = \int_0^\infty x e^{-px} dx = \frac{1}{p^2}. \tag{30}$$

$$\mathcal{L}_{e^{-\gamma x}} = \int_0^\infty e^{-\gamma x} e^{-px} dx = \frac{1}{p+\gamma}. \tag{31}$$

Noting that

$$-\frac{F_0}{p^2(\gamma+p)} = -\frac{F_0}{\gamma^2}\left(\frac{1}{\gamma+p} - \frac{1}{p} + \frac{\gamma}{p^2}\right), \tag{32}$$

we see that the inverse Laplace transform gives us

$$\phi(x) = \frac{F_0}{\gamma^2}\left(1 - e^{-\gamma x} - \gamma x\right). \tag{33}$$

We can check that this a solution to the differential equation

$$-\frac{d^2\phi}{dx^2} = F_0 e^{-\gamma x} \quad \text{for} \quad \phi(0) = 0 \quad \text{and} \quad \frac{d\phi}{dx}(0) = 0$$

Using Laplace transforms to solve equation :

$$\left(-\frac{d^2}{dx^2} - 1\right)\phi(x) = F_0 \sin\left(\frac{\pi x}{L}\right) \quad \text{with} \quad \phi(0) = 0,\ \frac{d\phi(0)}{dx} = 0$$

$$\mathcal{L}_\phi(p) = -\left(\frac{\pi}{L}\right)\frac{F_0}{\left(p^2+1\right)\left(p^2 + \left(\frac{\pi}{L}\right)^2\right)}$$

$$= -F_0\left(\frac{\pi/L}{(\pi/L)^2 - 1}\right)\left(\frac{1}{p^2+1} - \frac{1}{p^2 + \left(\frac{\pi}{L}\right)^2}\right)$$

Note that : $\int_0^\infty \sin(at) e^{-pt} dt = \frac{a}{a^2+p^2}$

$$\Rightarrow \phi(x) = \frac{F_0}{(\pi/L)^2 - 1}\left(\sin\left(\frac{\pi x}{L}\right) - \frac{\pi}{L}\sin(x)\right)$$

Inverse Laplace transform :

$$\mathcal{L}_\phi(p) = \int_0^\infty e^{-pt}\phi(t)dt$$

$$\phi(t) = \frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty} e^{pt}\mathcal{L}_\phi(p)dp$$

Check: $\frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty} e^{pt}\mathcal{L}_\varphi(p)dp = \frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty} e^{pt}dp\int_0^\infty e^{-pu}\varphi(u)du$

$$\frac{1}{2\pi i}\int_0^\infty \varphi(u)du\int_{\lambda-i\infty}^{\lambda+i\infty} e^{p(t-u)}dp = \frac{1}{2\pi i}\int_0^\infty \varphi(u)du\int_{-\infty}^{\infty} e^{\lambda(t-u)}e^{is(t-u)}ids$$

$$= \frac{1}{2\pi i}\int_0^\infty \varphi(u)du\left(e^{\lambda(t-u)}2\pi i\ \delta(t-u)\right)$$

$$= \begin{cases} \varphi(t) & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Complex numbers

$$i \equiv \sqrt{-1} \qquad i^2 = -1$$

Define $z = x + iy$

$$|z|^2 = zz^* = (x+iy)(x-iy) = x^2 + y^2$$

Polar representation

$$z = \rho(\cos\phi + i\sin\phi) = \rho\ e^{i\phi}$$

Functions of complex variables

$$f(z) = \Re(f(z)) + i\Im(f(z)) \equiv u(x,y) + iv(x,y)$$

Derivatives: Cauchy-Riemann equations

$$\frac{\partial f(z)}{\partial x} = \frac{\partial u(z)}{\partial x} + i\frac{\partial v(z)}{\partial x} \qquad \frac{\partial f(z)}{i\partial y} = \frac{\partial u(z)}{i\partial y} + i\frac{\partial v(z)}{i\partial y} = \frac{\partial v(z)}{\partial y} - i\frac{\partial u(z)}{\partial y}$$

Argue that $\frac{df}{dz} = \frac{\partial f(z)}{\partial x} = \frac{\partial f(z)}{i\partial y} \quad \Rightarrow \frac{\partial u(z)}{\partial x} = \frac{\partial v(z)}{\partial y}$ and $\frac{\partial v(z)}{\partial x} = -\frac{\partial u(z)}{\partial y}$

Analytic function

$f(z)$ is analytic if it is:

- continuous
- single valued
- its first derivative satisfies Cauchy-Rieman conditions

Which of the following functions are analytic?

$f(z) = e^z$

$f(z) = z^n$

$f(z) = \ln z$

$f(z) = z^\alpha$

Some details

$$e^z = e^{x+iy} = e^x \cos(y) + ie^x \sin(y)$$

$$\frac{\partial u}{\partial x} = e^x \cos(y) = \frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} = e^x \sin(y) = -\frac{\partial u}{\partial y}$$

$$z^2 = (x+iy)^2 = (x^2 - y^2) + 2ixy$$

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} = 2y = -\frac{\partial u}{\partial y}$$

PHY 711 – Contour Integration

These notes summarize some basic properties of complex functions and their integrals. An *analytic* function $f(z)$ in a certain region of the complex plane z is one which takes a single (non-infinite) value and is differentiable within that region. Cauchy's theorm states that a closed contour integral of the function within that region has the value

$$\oint_C f(z) = 0. \qquad (1)$$

As an example, functions composed of integer powers of z –

$$f(z) = z^n, \quad \text{for} \quad n = 0, 1, \pm 2, \pm 3.... \qquad (2)$$

fall in this catogory. Notice that non-integral powers are generally not analytic and that $n = -1$ is also special. In fact, we can show that

$$\oint_C \frac{dz}{z} = 2\pi i. \qquad (3)$$

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This result follows from the fact that we can deform the contour to a unit circle about the origin so that $z = e^{i\theta}$. Then

$$\oint_C \frac{dz}{z} = \int_0^{2\pi} \frac{e^{i\theta}}{e^{i\theta}} i d\theta = 2\pi i. \qquad (4)$$

Another result of this analysis is the Residue Theorm which states that if the complex function $g(z)$ has poles at a finite number of points z_p within a region C but is otherwise analytic, the contour integral can be avaluated according to

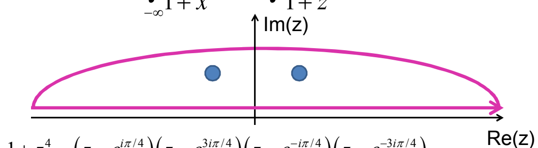
$$\oint_C g(z)dz = 2\pi i \sum_p Res(g_p), \tag{6}$$

where the residue is given by

$$Res(g_p) \equiv \lim_{z\to z_p} \left\{ \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left((z-z_p)^m g(z)\right) \right\}, \tag{7}$$

where m denotes the order of the pole.

Example: $\int_{-\infty}^{\infty} \frac{x^2}{1+x^4}dx = \oint \frac{z^2}{1+z^4}dz$



$$1+z^4 = \left(z-e^{i\pi/4}\right)\left(z-e^{3i\pi/4}\right)\left(z-e^{-i\pi/4}\right)\left(z-e^{-3i\pi/4}\right)$$

$$\oint \frac{z^2}{1+z^4}dz = 2\pi i\left(\mathrm{Res}\left(z_p = e^{i\pi/4}\right) + \mathrm{Res}\left(z_p = e^{3i\pi/4}\right)\right)$$

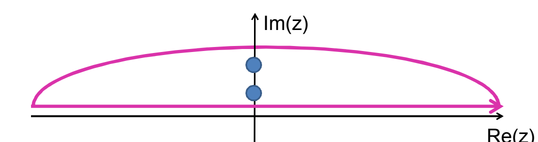
$$\mathrm{Res}\left(z_p = e^{i\pi/4}\right) = \frac{e^{i\pi/4}}{4i} \qquad \mathrm{Res}\left(z_p = e^{3i\pi/4}\right) = -\frac{e^{3i\pi/4}}{4i}$$

$$\oint \frac{z^2}{1+z^4}dz = 2\pi i\left(\frac{e^{i\pi/4}}{4i} - \frac{e^{3i\pi/4}}{4i}\right) = \frac{\pi}{\sqrt{2}}$$

Another example: $I = \int_0^\infty \frac{\cos(ax)}{4x^4+5x^2+1}dx.$

$$\int_0^\infty \frac{\cos(ax)}{4x^4+5x^2+1}dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{e^{iax}}{4x^4+5x^2+1}dx = \frac{1}{2}\oint \frac{e^{iaz}}{4z^4+5z^2+1}dz$$

$$4z^4+5z^2+1 = 4(z-i)\left(z-\tfrac{i}{2}\right)(z+i)\left(z+\tfrac{i}{2}\right)$$



$$I = 2\pi i\left(\mathrm{Res}\left(z_p = i\right) + \mathrm{Res}\left(z_p = \tfrac{i}{2}\right)\right)$$

$$\int_0^\infty \frac{\cos(ax)}{4x^4+5x^2+1}dx = \frac{1}{2}\oint \frac{e^{iaz}}{4z^4+5z^2+1}dz$$
$$= 2\pi i\left(\text{Res}\left(z_p = i\right) + \text{Res}\left(z_p = \tfrac{i}{2}\right)\right)$$
$$= \frac{\pi}{6}\left(-e^{-a} + 2e^{-a/2}\right)$$

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Cauchy integral theorem for analytic function *f(z)*:

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(z')}{z'-z}dz'.$$

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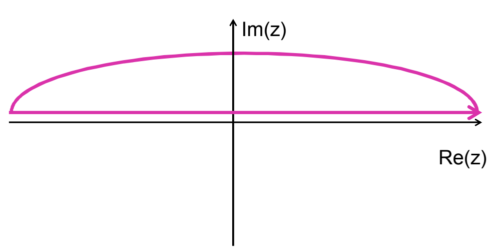
Example

Suppose $f\left(|z|\to\infty\right)=0$ and for $z=x$:

$$f(x) = a(x) + ib(x)$$



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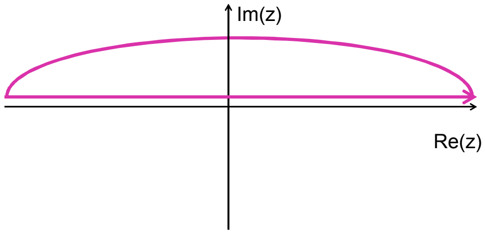
Example -- continued

$$f(z)=\frac{1}{2\pi i}\oint\frac{f(z')}{z'-z}dz' \quad \text{where} \quad f(x)=a(x)+ib(x)$$



$$a(x)+ib(x)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{a(x')+ib(x')}{x'-x}dx'$$

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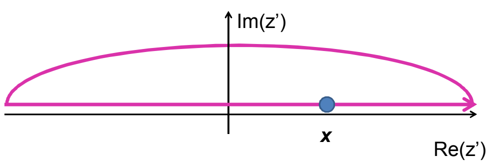
Example -- continued



$$\int_{-\infty}^{\infty}\frac{f(x')}{x'-x}dx'=\int_{-\infty}^{x-\varepsilon}\frac{f(x')}{x'-x}dx'+\int_{x+\varepsilon}^{\infty}\frac{f(x')}{x'-x}dx'+\int_{x-\varepsilon}^{x+\varepsilon}\frac{f(x')}{x'-x}dx'$$

$$=P\int_{-\infty}^{\infty}\frac{f(x')}{x'-x}dx'+i\pi f(x)$$

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Example -- continued

$$\int_{-\infty}^{\infty}\frac{f(x')}{x'-x}dx'=\int_{-\infty}^{x-\varepsilon}\frac{f(x')}{x'-x}dx'+\int_{x+\varepsilon}^{\infty}\frac{f(x')}{x'-x}dx'+\int_{x-\varepsilon}^{x+\varepsilon}\frac{f(x')}{x'-x}dx'$$

$$=P\int_{-\infty}^{\infty}\frac{f(x')}{x'-x}dx'+i\pi f(x)$$

$$a(x)+ib(x)=\frac{P}{2\pi i}\int_{-\infty}^{\infty}\frac{a(x')+ib(x')}{x'-x}dx'+\frac{\pi i}{2\pi i}\left(a(x)+ib(x)\right)$$

$$\Rightarrow a(x)=\frac{P}{\pi}\int_{-\infty}^{\infty}\frac{b(x')}{x'-x}dx' \qquad b(x)=-\frac{P}{\pi}\int_{-\infty}^{\infty}\frac{a(x')}{x'-x}dx'$$

Kramers-Kronig relationships

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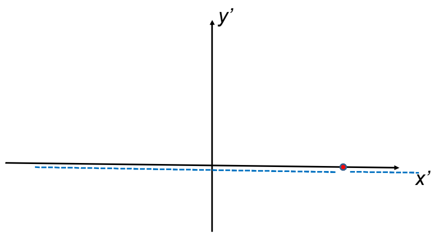
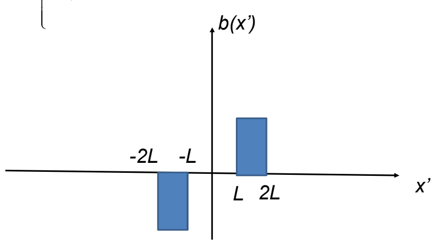
Comment on evaluating principal parts integrals

$$a(x) = \frac{P}{\pi}\int_{-\infty}^{\infty}\frac{b(x')}{x'-x}dx' = \lim_{\epsilon\to 0}\left(\frac{1}{\pi}\int_{-\infty}^{x-\epsilon}\frac{b(x')}{x'-x}dx' + \frac{1}{\pi}\int_{x+\epsilon}^{\infty}\frac{b(x')}{x'-x}dx'\right)$$



Example:

$$b(x') = \begin{cases} 0 & \text{for } x' < -2L, \quad -L < x' < L, \quad x' > 2L \\ B_0 & \text{for } L < x' < 2L \\ -B_0 & \text{for } -2L < x' < -L \end{cases}$$



$$a(x) = \frac{P}{\pi}\int_{-\infty}^{\infty}\frac{b(x')}{x'-x}dx' = \lim_{\epsilon\to 0}\left(\frac{1}{\pi}\int_{-\infty}^{x-\epsilon}\frac{b(x')}{x'-x}dx' + \frac{1}{\pi}\int_{x+\epsilon}^{\infty}\frac{b(x')}{x'-x}dx'\right)$$

For our example: $$a(x) = \frac{B_0}{\pi}\ln\left(\left|\frac{4L^2 - x^2}{L^2 - x^2}\right|\right)$$