

**PHY 711 Classical Mechanics and
Mathematical Methods
11-11:50 AM MWF Olin 107**

Plan for Lecture 20:

Rotational motion (Chapter 5)

- 1. Rigid body motion**
- 2. Moment of inertia tensor**
- 3. Torque free motion**

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5	Mon, 9/12/2016	Chap. 3	Calculus of variations	#5	9/14/2016
6	Wed, 9/14/2016	Chap. 3	Calculus of variations	#6	9/16/2016
7	Fri, 9/16/2016	Chap. 3	Lagrangian mechanics	#7	9/19/2016
8	Mon, 9/19/2016	Chap. 3 and 6	Lagrangian mechanics and constraints	#8	9/21/2016
9	Wed, 9/21/2016	Chap. 3 and 6	Constants of the motion	#9	9/23/2016
10	Fri, 9/23/2016	Chap. 3 and 6	Hamiltonian and canonical equations of motion	#10	9/26/2016
11	Mon, 9/26/2016	Chap. 3 and 6	Phase space	#11	9/28/2016
12	Wed, 9/28/2016	Chap. 6	Canonical transformations	#12	9/30/2016
13	Fri, 9/30/2016	Chap. 4	Small oscillations	#13	10/04/2016
14	Tue, 10/04/2016	Chap. 4	Normal modes	#14	10/07/2016
15	Wed, 10/05/2016	Chap. 7	Wave motion in one dimension	#15	10/07/2016
16	Fri, 10/07/2016	Chap. 7	Sturm-Liouville equations		
17	Mon, 10/10/2016	Chap. 7	Sturm-Liouville equations	Take-home exam	
18	Wed, 10/12/2016	Chap. 7	Fourier series and transforms	Take-home exam	
19	Fri, 10/14/2016	App. A	Laplace transforms and contour integrals	Take-home exam	
20	Mon, 10/17/2016	Chap. 5	Mechanics of rigid bodies	Exam due	
21	Wed, 10/19/2016	Chap. 5	Mechanics of rigid bodies	#16	10/24/2016
	Fri, 10/21/2016		Fall break -- no class		
22	Mon, 10/24/2016				
23	Wed, 11/02/2016				
24	Fri, 11/04/2016				

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Comparison of analysis in "inertial frame" versus "non-inertial frame"

Denote by \hat{e}_i^0 a fixed coordinate system

Denote by \hat{e}_i a moving coordinate system

For an arbitrary vector \mathbf{V} : $\mathbf{V} = \sum_{i=1}^3 V_i^0 \hat{e}_i^0 = \sum_{i=1}^3 V_i \hat{e}_i$

$$\left(\frac{d\mathbf{V}}{dt}\right)_{inertial} = \sum_{i=1}^3 \frac{dV_i^0}{dt} \hat{e}_i^0 = \sum_{i=1}^3 \frac{dV_i}{dt} \hat{e}_i + \sum_{i=1}^3 V_i \frac{d\hat{e}_i}{dt}$$

Define: $\left(\frac{d\mathbf{V}}{dt}\right)_{body} \equiv \sum_{i=1}^3 \frac{dV_i}{dt} \hat{e}_i$

$$\Rightarrow \left(\frac{d\mathbf{V}}{dt}\right)_{inertial} = \left(\frac{d\mathbf{V}}{dt}\right)_{body} + \sum_{i=1}^3 V_i \frac{d\hat{e}_i}{dt}$$

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The physics of rigid body motion; body fixed frame vs inertial frame; results from Chapter 2:

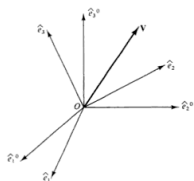


Figure 6.1 Transformation to a rotating coordinate system.

Let \mathbf{V} be a general vector, e.g., the position of a particle. This vector can be characterized by its components with respect to either orthonormal triad. Thus we can write

$$\mathbf{V} = \sum_{i=1}^3 V_i \hat{e}_i \quad (6.1a)$$

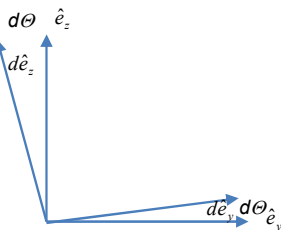
$$\mathbf{V} = \sum_{i=1}^3 V'_i \hat{e}'_i \quad (6.1b)$$

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Properties of the frame motion (rotation):



$$\begin{aligned} d\hat{e}_y &= d\Theta \hat{e}_z \\ d\hat{e}_z &= -d\Theta \hat{e}_y \\ \Rightarrow d\hat{\mathbf{e}} &= d\Theta \times \hat{\mathbf{e}} \\ \frac{d\hat{\mathbf{e}}}{dt} &= \frac{d\Theta}{dt} \times \hat{\mathbf{e}} \\ \frac{d\hat{\mathbf{e}}}{dt} &= \boldsymbol{\omega} \times \hat{\mathbf{e}} \end{aligned}$$

$$\begin{pmatrix} d\hat{e}_y \\ d\hat{e}_z \end{pmatrix} = \begin{pmatrix} \cos(d\Theta) & \sin(d\Theta) \\ -\sin(d\Theta) & \cos(d\Theta) \end{pmatrix} \begin{pmatrix} \hat{e}_y \\ \hat{e}_z \end{pmatrix} - \begin{pmatrix} \hat{e}_y \\ \hat{e}_z \end{pmatrix}$$

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$$\begin{aligned} \left(\frac{d\mathbf{V}}{dt}\right)_{inertial} &= \left(\frac{d\mathbf{V}}{dt}\right)_{body} + \sum_{i=1}^3 V_i \frac{d\hat{e}_i}{dt} \\ \left(\frac{d\mathbf{V}}{dt}\right)_{inertial} &= \left(\frac{d\mathbf{V}}{dt}\right)_{body} + \boldsymbol{\omega} \times \mathbf{V} \end{aligned}$$

Effects on acceleration:

$$\begin{aligned} \left(\frac{d}{dt} \frac{d\mathbf{V}}{dt}\right)_{inertial} &= \left(\left(\frac{d}{dt}\right)_{body} + \boldsymbol{\omega} \times\right) \left\{ \left(\frac{d\mathbf{V}}{dt}\right)_{body} + \boldsymbol{\omega} \times \mathbf{V} \right\} \\ \left(\frac{d^2\mathbf{V}}{dt^2}\right)_{inertial} &= \left(\frac{d^2\mathbf{V}}{dt^2}\right)_{body} + 2\boldsymbol{\omega} \times \left(\frac{d\mathbf{V}}{dt}\right)_{body} + \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{V} + \boldsymbol{\omega} \times \boldsymbol{\omega} \times \mathbf{V} \end{aligned}$$

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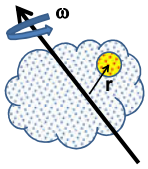
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Kinetic energy of rigid body :

$$\left(\frac{d\mathbf{r}}{dt}\right)_{inertial} = \left(\frac{d\mathbf{r}}{dt}\right)_{body} + \boldsymbol{\omega} \times \mathbf{r}$$

$\left(\frac{d\mathbf{r}}{dt}\right)_{body} = 0$ for rigid body

$$\rightarrow \left(\frac{d\mathbf{r}}{dt}\right)_{inertial} = \boldsymbol{\omega} \times \mathbf{r}$$


$$T = \sum_p \frac{1}{2} m_p v_p^2 = \sum_p \frac{1}{2} m_p (\boldsymbol{\omega} \times \mathbf{r}_p)^2$$

$$= \sum_p \frac{1}{2} m_p (\boldsymbol{\omega} \times \mathbf{r}_p) \cdot (\boldsymbol{\omega} \times \mathbf{r}_p)$$

$$= \sum_p \frac{1}{2} m_p [(\boldsymbol{\omega} \cdot \boldsymbol{\omega})(\mathbf{r}_p \cdot \mathbf{r}_p) - (\mathbf{r}_p \cdot \boldsymbol{\omega})^2]$$

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$$T = \sum_p \frac{1}{2} m_p [(\boldsymbol{\omega} \cdot \boldsymbol{\omega})(\mathbf{r}_p \cdot \mathbf{r}_p) - (\mathbf{r}_p \cdot \boldsymbol{\omega})^2]$$

$$= \frac{1}{2} \boldsymbol{\omega} \cdot \bar{\mathbf{I}} \cdot \boldsymbol{\omega}$$

Moment of inertia tensor :

$$\bar{\mathbf{I}} \equiv \sum_p m_p (\mathbf{r}_p^2 \mathbf{1} - \mathbf{r}_p \mathbf{r}_p)$$
 (dyad notation)

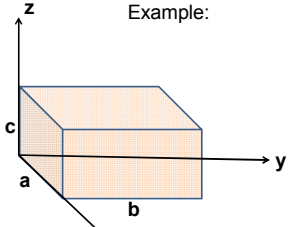
Matrix notation :

$$\bar{\mathbf{I}} \equiv \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$$

$$I_{ij} \equiv \sum_p m_p (\delta_{ij} r_p^2 - r_{pi} r_{pj})$$

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Example:



Moment of inertia tensor :

$$\bar{\mathbf{I}} = M \begin{pmatrix} \frac{1}{3}(b^2 + c^2) & -\frac{1}{4}ab & -\frac{1}{4}ac \\ -\frac{1}{4}ab & \frac{1}{3}(a^2 + c^2) & -\frac{1}{4}bc \\ -\frac{1}{4}ac & -\frac{1}{4}bc & \frac{1}{3}(a^2 + b^2) \end{pmatrix}$$

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Properties of moment of inertia tensor:

- > Symmetric matrix → real eigenvalues I_1, I_2, I_3
- > → orthogonal eigenvectors

$$\vec{I} \cdot \hat{e}_i = I_i \hat{e}_i \quad i = 1, 2, 3$$

Moment of inertia tensor :

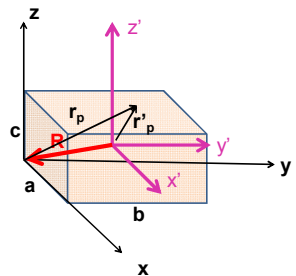
$$\vec{I} = M \begin{pmatrix} \frac{1}{3}(b^2 + c^2) & -\frac{1}{4}ab & -\frac{1}{4}ac \\ -\frac{1}{4}ab & \frac{1}{3}(a^2 + c^2) & -\frac{1}{4}bc \\ -\frac{1}{4}ac & -\frac{1}{4}bc & \frac{1}{3}(a^2 + b^2) \end{pmatrix}$$

For $a = b = c$:

$$I_1 = \frac{1}{6}Ma^2 \quad I_2 = \frac{11}{12}Ma^2 \quad I_3 = \frac{11}{12}Ma^2$$

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Changing origin of rotation



$$I_{ij} \equiv \sum_p m_p (\delta_{ij} r_p^2 - r_{pi} r_{pj})$$

$$I'_{ij} \equiv \sum_p m_p (\delta_{ij} r_p'^2 - r'_{pi} r'_{pj})$$

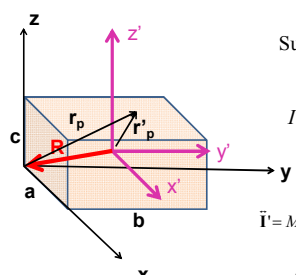
$$\mathbf{r}'_p = \mathbf{r}_p + \mathbf{R}$$

Define the center of mass :

$$\mathbf{r}_{CM} = \frac{\sum_p m_p \mathbf{r}_p}{\sum_p m_p} \equiv \frac{\sum_p m_p \mathbf{r}_p}{M}$$

$$I'_{ij} = I_{ij} + M(R^2 \delta_{ij} - R_i R_j) + M(2\mathbf{r}_{CM} \cdot \mathbf{R} \delta_{ij} - r_{CMi} R_j - R_i r_{CMj})$$

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$$I'_{ij} = I_{ij} + M(R^2 \delta_{ij} - R_i R_j) + M(2\mathbf{r}_{CM} \cdot \mathbf{R} \delta_{ij} - r_{CMi} R_j - R_i r_{CMj})$$


Suppose that $\mathbf{R} = -\frac{a}{2} \hat{x} - \frac{b}{2} \hat{y} - \frac{c}{2} \hat{z}$
and $\mathbf{r}_{CM} = -\mathbf{R}$

$$I'_{ij} = I_{ij} - M(R^2 \delta_{ij} - R_i R_j)$$

$$\vec{I} = M \begin{pmatrix} \frac{1}{3}(b^2 + c^2) & -\frac{1}{4}ab & -\frac{1}{4}ac \\ -\frac{1}{4}ab & \frac{1}{3}(a^2 + c^2) & -\frac{1}{4}bc \\ -\frac{1}{4}ac & -\frac{1}{4}bc & \frac{1}{3}(a^2 + b^2) \end{pmatrix}$$

$$- M \begin{pmatrix} \frac{1}{4}(b^2 + c^2) & -\frac{1}{4}ab & -\frac{1}{4}ac \\ -\frac{1}{4}ab & \frac{1}{4}(a^2 + c^2) & -\frac{1}{4}bc \\ -\frac{1}{4}ac & -\frac{1}{4}bc & \frac{1}{4}(a^2 + b^2) \end{pmatrix}$$

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Note: This is a special case; changing the center of rotation does not necessarily result in a diagonal I'

$$\bar{\mathbf{I}}' = M \begin{pmatrix} \frac{1}{12}(b^2 + c^2) & 0 & 0 \\ 0 & \frac{1}{12}(a^2 + c^2) & 0 \\ 0 & 0 & \frac{1}{12}(a^2 + b^2) \end{pmatrix}$$

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Descriptions of rotation about a given origin

For general coordinate system

$$T = \frac{1}{2} \sum_{ij} I_{ij} \omega_i \omega_j$$

For (body fixed) coordinate system that diagonalizes moment of inertia tensor :

$$\bar{\mathbf{I}} \cdot \hat{\mathbf{e}}_i = I_i \hat{\mathbf{e}}_i \quad i = 1, 2, 3$$

$$\boldsymbol{\omega} = \tilde{\omega}_1 \hat{\mathbf{e}}_1 + \tilde{\omega}_2 \hat{\mathbf{e}}_2 + \tilde{\omega}_3 \hat{\mathbf{e}}_3$$

$$\Rightarrow T = \frac{1}{2} \sum_i I_i \tilde{\omega}_i^2$$

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Descriptions of rotation about a given origin -- continued

Time rate of change of angular momentum

$$\frac{d\mathbf{L}}{dt} = \left(\frac{d\mathbf{L}}{dt} \right)_{body} + \boldsymbol{\omega} \times \mathbf{L}$$

For (body fixed) coordinate system that diagonalizes moment of inertia tensor :

$$\bar{\mathbf{I}} \cdot \hat{\mathbf{e}}_i = I_i \hat{\mathbf{e}}_i \quad \boldsymbol{\omega} = \tilde{\omega}_1 \hat{\mathbf{e}}_1 + \tilde{\omega}_2 \hat{\mathbf{e}}_2 + \tilde{\omega}_3 \hat{\mathbf{e}}_3$$

$$\mathbf{L} = I_1 \tilde{\omega}_1 \hat{\mathbf{e}}_1 + I_2 \tilde{\omega}_2 \hat{\mathbf{e}}_2 + I_3 \tilde{\omega}_3 \hat{\mathbf{e}}_3$$

$$\frac{d\mathbf{L}}{dt} = I_1 \dot{\tilde{\omega}}_1 \hat{\mathbf{e}}_1 + I_2 \dot{\tilde{\omega}}_2 \hat{\mathbf{e}}_2 + I_3 \dot{\tilde{\omega}}_3 \hat{\mathbf{e}}_3 + \tilde{\omega}_2 \tilde{\omega}_3 (I_3 - I_2) \hat{\mathbf{e}}_1 + \tilde{\omega}_3 \tilde{\omega}_1 (I_1 - I_3) \hat{\mathbf{e}}_2 + \tilde{\omega}_1 \tilde{\omega}_2 (I_2 - I_1) \hat{\mathbf{e}}_3$$

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Descriptions of rotation about a given origin -- continued

Note that the torque equation

$$\frac{d\mathbf{L}}{dt} = \left(\frac{d\mathbf{L}}{dt} \right)_{body} + \boldsymbol{\omega} \times \mathbf{L} = \boldsymbol{\tau}$$

is very difficult to solve directly in the body fixed frame.

For $\boldsymbol{\tau} = 0$ we can solve the Euler equations :

$$\begin{aligned} \frac{d\mathbf{L}}{dt} = I_1 \dot{\tilde{\omega}}_1 \hat{\mathbf{e}}_1 + I_2 \dot{\tilde{\omega}}_2 \hat{\mathbf{e}}_2 + I_3 \dot{\tilde{\omega}}_3 \hat{\mathbf{e}}_3 + \tilde{\omega}_2 \tilde{\omega}_3 (I_3 - I_2) \hat{\mathbf{e}}_1 \\ + \tilde{\omega}_3 \tilde{\omega}_1 (I_1 - I_3) \hat{\mathbf{e}}_2 + \tilde{\omega}_1 \tilde{\omega}_2 (I_2 - I_1) \hat{\mathbf{e}}_3 = 0 \end{aligned}$$

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Euler equations for rotation in body fixed frame :

$$I_1 \dot{\tilde{\omega}}_1 + \tilde{\omega}_2 \tilde{\omega}_3 (I_3 - I_2) = 0$$

$$I_2 \dot{\tilde{\omega}}_2 + \tilde{\omega}_3 \tilde{\omega}_1 (I_1 - I_3) = 0$$

$$I_3 \dot{\tilde{\omega}}_3 + \tilde{\omega}_1 \tilde{\omega}_2 (I_2 - I_1) = 0$$

→ Solution for symmetric top -- $I_2 = I_1$:

$$I_1 \dot{\tilde{\omega}}_1 + \tilde{\omega}_2 \tilde{\omega}_3 (I_3 - I_1) = 0$$

$$I_1 \dot{\tilde{\omega}}_2 + \tilde{\omega}_3 \tilde{\omega}_1 (I_1 - I_3) = 0$$

$$I_3 \dot{\tilde{\omega}}_3 = 0 \quad \Rightarrow \quad \tilde{\omega}_3 = (\text{constant})$$

$$\text{Define : } \quad \Omega \equiv \tilde{\omega}_3 \frac{I_3 - I_1}{I_1} \quad \begin{aligned} \dot{\tilde{\omega}}_1 &= -\tilde{\omega}_2 \Omega \\ \dot{\tilde{\omega}}_2 &= \tilde{\omega}_1 \Omega \end{aligned}$$

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Solution of Euler equations for a symmetric top -- continued

$$\dot{\tilde{\omega}}_1 = -\tilde{\omega}_2 \Omega \quad \dot{\tilde{\omega}}_2 = \tilde{\omega}_1 \Omega$$

$$\text{where } \Omega \equiv \tilde{\omega}_3 \frac{I_3 - I_1}{I_1}$$

$$\text{Solution: } \quad \tilde{\omega}_1(t) = A \cos(\Omega t + \phi)$$

$$\tilde{\omega}_2(t) = A \sin(\Omega t + \phi)$$

$$\tilde{\omega}_3(t) = \tilde{\omega}_3 \quad (\text{constant})$$

$$T = \frac{1}{2} \sum_i I_i \tilde{\omega}_i^2 = \frac{1}{2} I_1 A^2 + \frac{1}{2} I_3 \tilde{\omega}_3^2$$

$$\begin{aligned} \mathbf{L} &= I_1 \tilde{\omega}_1 \hat{\mathbf{e}}_1 + I_2 \tilde{\omega}_2 \hat{\mathbf{e}}_2 + I_3 \tilde{\omega}_3 \hat{\mathbf{e}}_3 \\ &= I_1 A (\cos(\Omega t + \phi) \hat{\mathbf{e}}_1 + \sin(\Omega t + \phi) \hat{\mathbf{e}}_2) + I_3 \tilde{\omega}_3 \hat{\mathbf{e}}_3 \end{aligned}$$

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Euler equations for rotation in body fixed frame :

$$I_1 \dot{\tilde{\omega}}_1 + \tilde{\omega}_2 \tilde{\omega}_3 (I_3 - I_2) = 0$$

$$I_2 \dot{\tilde{\omega}}_2 + \tilde{\omega}_3 \tilde{\omega}_1 (I_1 - I_3) = 0$$

$$I_3 \dot{\tilde{\omega}}_3 + \tilde{\omega}_1 \tilde{\omega}_2 (I_2 - I_1) = 0$$

→ Solution for asymmetric top -- $I_3 \neq I_2 \neq I_1$:

$$I_1 \dot{\tilde{\omega}}_1 + \tilde{\omega}_2 \tilde{\omega}_3 (I_3 - I_2) = 0$$

$$I_2 \dot{\tilde{\omega}}_2 + \tilde{\omega}_3 \tilde{\omega}_1 (I_1 - I_3) = 0$$

$$I_3 \dot{\tilde{\omega}}_3 + \tilde{\omega}_1 \tilde{\omega}_2 (I_2 - I_1) = 0$$

Suppose : $\dot{\tilde{\omega}}_3 \approx 0$ Define : $\Omega_1 \equiv \tilde{\omega}_3 \frac{I_3 - I_2}{I_1}$

Define : $\Omega_2 \equiv \tilde{\omega}_3 \frac{I_3 - I_1}{I_2}$

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Euler equations for rotation in body fixed frame :

$$I_1 \dot{\tilde{\omega}}_1 + \tilde{\omega}_2 \tilde{\omega}_3 (I_3 - I_2) = 0$$

$$I_2 \dot{\tilde{\omega}}_2 + \tilde{\omega}_3 \tilde{\omega}_1 (I_1 - I_3) = 0$$

$$I_3 \dot{\tilde{\omega}}_3 + \tilde{\omega}_1 \tilde{\omega}_2 (I_2 - I_1) = 0$$

Solution for asymmetric top -- $I_3 \neq I_2 \neq I_1$:

Approximate solution --

Suppose: $\dot{\tilde{\omega}}_3 \approx 0$ Define: $\Omega_1 \equiv \tilde{\omega}_3 \frac{I_3 - I_2}{I_1}$

Define: $\Omega_2 \equiv \tilde{\omega}_3 \frac{I_3 - I_1}{I_2}$

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Euler equations for asymmetric top -- continued

$$I_1 \dot{\tilde{\omega}}_1 + \tilde{\omega}_2 \tilde{\omega}_3 (I_3 - I_2) = 0$$

$$I_2 \dot{\tilde{\omega}}_2 + \tilde{\omega}_3 \tilde{\omega}_1 (I_1 - I_3) = 0$$

$$I_3 \dot{\tilde{\omega}}_3 + \tilde{\omega}_1 \tilde{\omega}_2 (I_2 - I_1) = 0$$

If $\dot{\tilde{\omega}}_3 \approx 0$, Define: $\Omega_1 \equiv \tilde{\omega}_3 \frac{I_3 - I_2}{I_1}$ $\Omega_2 \equiv \tilde{\omega}_3 \frac{I_3 - I_1}{I_2}$

$$\dot{\tilde{\omega}}_1 = -\Omega_1 \tilde{\omega}_2 \quad \dot{\tilde{\omega}}_2 = \Omega_2 \tilde{\omega}_1$$

If Ω_1 and Ω_2 are both positive or both negative :

$$\tilde{\omega}_1(t) \approx A \cos(\sqrt{\Omega_1 \Omega_2} t + \varphi)$$

$$\tilde{\omega}_2(t) \approx A \sqrt{\frac{\Omega_2}{\Omega_1}} \sin(\sqrt{\Omega_1 \Omega_2} t + \varphi)$$

⇒ If Ω_1 and Ω_2 have opposite signs, solution is unstable.

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