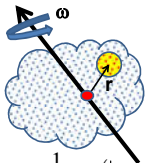




Summary of previous results describing rigid bodies rotating about a fixed origin ●



$$\left(\frac{d\mathbf{r}}{dt}\right)_{inertial} = \boldsymbol{\omega} \times \mathbf{r}$$

Kinetic energy:  $T = \sum_p \frac{1}{2} m_p v_p^2 = \sum_p \frac{1}{2} m_p (\boldsymbol{\omega} \times \mathbf{r}_p)^2$

$$= \sum_p \frac{1}{2} m_p (\boldsymbol{\omega} \times \mathbf{r}_p) \cdot (\boldsymbol{\omega} \times \mathbf{r}_p)$$

$$= \sum_p \frac{1}{2} m_p [(\boldsymbol{\omega} \cdot \boldsymbol{\omega})(\mathbf{r}_p \cdot \mathbf{r}_p) - (\mathbf{r}_p \cdot \boldsymbol{\omega})^2]$$

$$= \boldsymbol{\omega} \cdot \tilde{\mathbf{I}} \cdot \boldsymbol{\omega} \quad \tilde{\mathbf{I}} \equiv \sum_p m_p (\mathbf{1}r_p^2 - \mathbf{r}_p \mathbf{r}_p)$$

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Moment of inertia tensor  
Matrix notation:

$$\tilde{\mathbf{I}} \equiv \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \quad I_{ij} \equiv \sum_p m_p (\delta_{ij} r_p^2 - r_{pi} r_{pj})$$

For general coordinate system:  $T = \frac{1}{2} \sum_{ij} I_{ij} \omega_i \omega_j$

For (body fixed) coordinate system that diagonalizes moment of inertia tensor:  $\tilde{\mathbf{I}} \cdot \hat{\mathbf{e}}_i = I_i \hat{\mathbf{e}}_i \quad i = 1, 2, 3$

$$\boldsymbol{\omega} = \tilde{\omega}_1 \hat{\mathbf{e}}_1 + \tilde{\omega}_2 \hat{\mathbf{e}}_2 + \tilde{\omega}_3 \hat{\mathbf{e}}_3 \quad \Rightarrow T = \frac{1}{2} \sum_i I_i \tilde{\omega}_i^2$$

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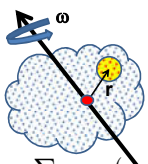
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Continued -- summary of previous results describing rigid bodies rotating about a fixed origin ●



$$\left(\frac{d\mathbf{r}}{dt}\right)_{inertial} = \boldsymbol{\omega} \times \mathbf{r}$$

Angular momentum:  $\mathbf{L} = \sum_p m_p \mathbf{r}_p \times \mathbf{v}_p = \sum_p m_p \mathbf{r}_p \times (\boldsymbol{\omega} \times \mathbf{r}_p)$

$$= \sum_p m_p [\boldsymbol{\omega} (\mathbf{r}_p \cdot \mathbf{r}_p) - \mathbf{r}_p (\mathbf{r}_p \cdot \boldsymbol{\omega})]$$

$$= \tilde{\mathbf{I}} \cdot \boldsymbol{\omega} \quad \tilde{\mathbf{I}} \equiv \sum_p m_p (\mathbf{1}r_p^2 - \mathbf{r}_p \mathbf{r}_p)$$

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Descriptions of rotation about a given origin -- continued

For (body fixed) coordinate system that diagonalizes moment of inertia tensor:

$$\vec{I} \cdot \hat{e}_i = I_i \hat{e}_i \quad \boldsymbol{\omega} = \tilde{\omega}_1 \hat{e}_1 + \tilde{\omega}_2 \hat{e}_2 + \tilde{\omega}_3 \hat{e}_3$$

$$\mathbf{L} = I_1 \tilde{\omega}_1 \hat{e}_1 + I_2 \tilde{\omega}_2 \hat{e}_2 + I_3 \tilde{\omega}_3 \hat{e}_3$$

Time derivative:  $\frac{d\mathbf{L}}{dt} = \left( \frac{d\mathbf{L}}{dt} \right)_{body} + \boldsymbol{\omega} \times \mathbf{L}$

$$\frac{d\mathbf{L}}{dt} = I_1 \dot{\tilde{\omega}}_1 \hat{e}_1 + I_2 \dot{\tilde{\omega}}_2 \hat{e}_2 + I_3 \dot{\tilde{\omega}}_3 \hat{e}_3 + \tilde{\omega}_2 \tilde{\omega}_3 (I_3 - I_2) \hat{e}_1 + \tilde{\omega}_3 \tilde{\omega}_1 (I_1 - I_3) \hat{e}_2 + \tilde{\omega}_1 \tilde{\omega}_2 (I_2 - I_1) \hat{e}_3$$

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Descriptions of rotation about a given origin -- continued

Note that the torque equation

$$\frac{d\mathbf{L}}{dt} = \left( \frac{d\mathbf{L}}{dt} \right)_{body} + \boldsymbol{\omega} \times \mathbf{L} = \boldsymbol{\tau}$$

is very difficult to solve directly in the body fixed frame.

For  $\boldsymbol{\tau} = 0$  we can solve the Euler equations:

$$\frac{d\mathbf{L}}{dt} = 0 = I_1 \dot{\tilde{\omega}}_1 \hat{e}_1 + I_2 \dot{\tilde{\omega}}_2 \hat{e}_2 + I_3 \dot{\tilde{\omega}}_3 \hat{e}_3 + \tilde{\omega}_2 \tilde{\omega}_3 (I_3 - I_2) \hat{e}_1 + \tilde{\omega}_3 \tilde{\omega}_1 (I_1 - I_3) \hat{e}_2 + \tilde{\omega}_1 \tilde{\omega}_2 (I_2 - I_1) \hat{e}_3$$

$$I_1 \dot{\tilde{\omega}}_1 + \tilde{\omega}_2 \tilde{\omega}_3 (I_3 - I_2) = 0$$

$$I_2 \dot{\tilde{\omega}}_2 + \tilde{\omega}_3 \tilde{\omega}_1 (I_1 - I_3) = 0$$

$$I_3 \dot{\tilde{\omega}}_3 + \tilde{\omega}_1 \tilde{\omega}_2 (I_2 - I_1) = 0$$

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Euler equations for rotation in body fixed frame :

$$I_1 \dot{\tilde{\omega}}_1 + \tilde{\omega}_2 \tilde{\omega}_3 (I_3 - I_2) = 0$$

$$I_2 \dot{\tilde{\omega}}_2 + \tilde{\omega}_3 \tilde{\omega}_1 (I_1 - I_3) = 0$$

$$I_3 \dot{\tilde{\omega}}_3 + \tilde{\omega}_1 \tilde{\omega}_2 (I_2 - I_1) = 0$$

Solution for symmetric top --  $I_2 = I_1$  :

$$I_1 \dot{\tilde{\omega}}_1 + \tilde{\omega}_2 \tilde{\omega}_3 (I_3 - I_1) = 0$$

$$I_1 \dot{\tilde{\omega}}_2 + \tilde{\omega}_3 \tilde{\omega}_1 (I_1 - I_3) = 0$$

$$I_3 \dot{\tilde{\omega}}_3 = 0 \quad \Rightarrow \quad \tilde{\omega}_3 = (\text{constant})$$

Define :  $\Omega \equiv \tilde{\omega}_3 \frac{I_3 - I_1}{I_1}$   $\dot{\tilde{\omega}}_1 = -\tilde{\omega}_2 \Omega$   
 $\dot{\tilde{\omega}}_2 = \tilde{\omega}_1 \Omega$

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Solution of Euler equations for a symmetric top -- continued

$$\dot{\tilde{\omega}}_1 = -\tilde{\omega}_2 \Omega \quad \dot{\tilde{\omega}}_2 = \tilde{\omega}_1 \Omega$$

where  $\Omega \equiv \tilde{\omega}_3 \frac{I_3 - I_1}{I_1}$

Solution :  $\tilde{\omega}_1(t) = A \cos(\Omega t + \varphi)$   
 $\tilde{\omega}_2(t) = A \sin(\Omega t + \varphi)$

$$T = \frac{1}{2} \sum_i I_i \tilde{\omega}_i^2 = \frac{1}{2} I_1 A^2 + \frac{1}{2} I_3 \tilde{\omega}_3^2$$

$\mathbf{L} = I_1 \tilde{\omega}_1 \hat{\mathbf{e}}_1 + I_2 \tilde{\omega}_2 \hat{\mathbf{e}}_2 + I_3 \tilde{\omega}_3 \hat{\mathbf{e}}_3$   
 $= I_1 A (\cos(\Omega t + \varphi) \hat{\mathbf{e}}_1 + \sin(\Omega t + \varphi) \hat{\mathbf{e}}_2) + I_3 \tilde{\omega}_3 \hat{\mathbf{e}}_3$

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Euler equations for rotation in body fixed frame :

$$I_1 \dot{\tilde{\omega}}_1 + \tilde{\omega}_2 \tilde{\omega}_3 (I_3 - I_2) = 0$$

$$I_2 \dot{\tilde{\omega}}_2 + \tilde{\omega}_3 \tilde{\omega}_1 (I_1 - I_3) = 0$$

$$I_3 \dot{\tilde{\omega}}_3 + \tilde{\omega}_1 \tilde{\omega}_2 (I_2 - I_1) = 0$$

Solution for asymmetric top --  $I_3 \neq I_2 \neq I_1$  :

Suppose :  $\dot{\tilde{\omega}}_3 \approx 0$  Define :  $\Omega_1 \equiv \tilde{\omega}_3 \frac{I_3 - I_2}{I_1}$   
 Define :  $\Omega_2 \equiv \tilde{\omega}_3 \frac{I_3 - I_1}{I_2}$

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Euler equations for asymmetric top -- continued

$$I_1 \dot{\tilde{\omega}}_1 + \tilde{\omega}_2 \tilde{\omega}_3 (I_3 - I_2) = 0$$

$$I_2 \dot{\tilde{\omega}}_2 + \tilde{\omega}_3 \tilde{\omega}_1 (I_1 - I_3) = 0$$

$$I_3 \dot{\tilde{\omega}}_3 + \tilde{\omega}_1 \tilde{\omega}_2 (I_2 - I_1) = 0$$

If  $\dot{\tilde{\omega}}_3 \approx 0$ , Define:  $\Omega_1 \equiv \tilde{\omega}_3 \frac{I_3 - I_2}{I_1}$   $\Omega_2 \equiv \tilde{\omega}_3 \frac{I_3 - I_1}{I_2}$

$$\dot{\tilde{\omega}}_1 = -\Omega_1 \tilde{\omega}_2 \quad \dot{\tilde{\omega}}_2 = \Omega_2 \tilde{\omega}_1$$

If  $\Omega_1$  and  $\Omega_2$  are both positive or both negative :

$$\tilde{\omega}_1(t) \approx A \cos(\sqrt{\Omega_1 \Omega_2} t + \varphi)$$

$$\tilde{\omega}_2(t) \approx A \sqrt{\frac{\Omega_2}{\Omega_1}} \sin(\sqrt{\Omega_1 \Omega_2} t + \varphi)$$

$\Rightarrow$  If  $\Omega_1$  and  $\Omega_2$  have opposite signs, solution is unstable.

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Transformation between body-fixed and inertial coordinate systems – Euler angles

[http://en.wikipedia.org/wiki/Euler\\_angles](http://en.wikipedia.org/wiki/Euler_angles)

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$$\tilde{\omega} = \alpha \hat{e}_3^0 + \beta \hat{e}_2' + \gamma \hat{e}_3$$

Need to express all components in body-fixed frame:

$$\tilde{\omega} = \tilde{\omega}_1 \hat{e}_1 + \tilde{\omega}_2 \hat{e}_2 + \tilde{\omega}_3 \hat{e}_3$$

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$$\tilde{\omega} = \alpha \hat{e}_3^0 + \beta \hat{e}_2' + \gamma \hat{e}_3$$

$$\hat{e}_2' = \sin \gamma \hat{e}_1 + \cos \gamma \hat{e}_2$$

Matrix representation :

$$\hat{e}_2' = \begin{pmatrix} \cos \gamma & \sin \gamma & 0 \\ -\sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \sin \gamma \\ \cos \gamma \\ 0 \end{pmatrix}$$

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$\tilde{\omega} = \dot{\alpha} \hat{e}_3^0 + \dot{\beta} \hat{e}_2 + \dot{\gamma} \hat{e}_3$

$\hat{e}_3^0 = -\sin \beta \hat{e}_1 + \cos \beta \hat{e}_3$   
 $= -\cos \gamma \sin \beta \hat{e}_1 + \sin \gamma \sin \beta \hat{e}_2 + \cos \beta \hat{e}_3$

Matrix representation:  
 $\hat{e}_3^0 = \begin{pmatrix} \cos \gamma & \sin \gamma & 0 \\ -\sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \beta & 0 & -\sin \beta \\ 0 & 1 & 0 \\ \sin \beta & 0 & \cos \beta \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$   
 $= \begin{pmatrix} -\sin \beta \cos \gamma \\ \sin \beta \sin \gamma \\ \cos \beta \end{pmatrix}$

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$\tilde{\omega} = \dot{\alpha} \hat{e}_3^0 + \dot{\beta} \hat{e}_2 + \dot{\gamma} \hat{e}_3$

$\tilde{\omega} = \dot{\alpha} \begin{pmatrix} -\sin \beta \cos \gamma \\ \sin \beta \sin \gamma \\ \cos \beta \end{pmatrix} + \dot{\beta} \begin{pmatrix} \sin \gamma \\ \cos \gamma \\ 0 \end{pmatrix} + \dot{\gamma} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\tilde{\omega} = \tilde{\omega}_1 \hat{e}_1 + \tilde{\omega}_2 \hat{e}_2 + \tilde{\omega}_3 \hat{e}_3$

$\tilde{\omega} = \dot{\alpha} \begin{pmatrix} -\sin \beta \cos \gamma \\ \sin \beta \sin \gamma \\ \cos \beta \end{pmatrix} + \dot{\beta} \begin{pmatrix} \sin \gamma \\ \cos \gamma \\ 0 \end{pmatrix} + \dot{\gamma} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\tilde{\omega}_1 = \dot{\alpha}(-\sin \beta \cos \gamma) + \dot{\beta} \sin \gamma$   
 $\tilde{\omega}_2 = \dot{\alpha}(\sin \beta \sin \gamma) + \dot{\beta} \cos \gamma$   
 $\tilde{\omega}_3 = \dot{\alpha} \cos \beta + \dot{\gamma}$

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$\tilde{\omega} = \dot{\alpha} \hat{e}_3^0 + \dot{\beta} \hat{e}_2 + \dot{\gamma} \hat{e}_3$

$\tilde{\omega} = [\dot{\alpha}(-\sin \beta \cos \gamma) + \dot{\beta} \sin \gamma] \hat{e}_1$   
 $+ [\dot{\alpha}(\sin \beta \sin \gamma) + \dot{\beta} \cos \gamma] \hat{e}_2$   
 $+ [\dot{\alpha} \cos \beta + \dot{\gamma}] \hat{e}_3$

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Rotational kinetic energy

$$\begin{aligned}
 T(\alpha, \beta, \gamma, \dot{\alpha}, \dot{\beta}, \dot{\gamma}) &= \frac{1}{2} I_1 \tilde{\omega}_1^2 + \frac{1}{2} I_2 \tilde{\omega}_2^2 + \frac{1}{2} I_3 \tilde{\omega}_3^2 \\
 &= \frac{1}{2} I_1 [\dot{\alpha}(-\sin \beta \cos \gamma) + \dot{\beta} \sin \gamma]^2 \\
 &\quad + \frac{1}{2} I_2 [\dot{\alpha}(\sin \beta \sin \gamma) + \dot{\beta} \cos \gamma]^2 \\
 &\quad + \frac{1}{2} I_3 [\dot{\alpha} \cos \beta + \dot{\gamma}]^2
 \end{aligned}$$

If  $I_1 = I_2$ :

$$T(\alpha, \beta, \gamma, \dot{\alpha}, \dot{\beta}, \dot{\gamma}) = \frac{1}{2} I_1 (\dot{\alpha}^2 \sin^2 \beta + \dot{\beta}^2) + \frac{1}{2} I_3 [\dot{\alpha} \cos \beta + \dot{\gamma}]^2$$

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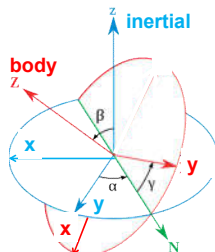
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Transformation between body-fixed and inertial coordinate systems – Euler angles



[http://en.wikipedia.org/wiki/Euler\\_angles](http://en.wikipedia.org/wiki/Euler_angles)

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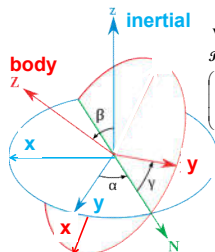
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General transformation between rotated coordinates – Euler angles



$$\mathbf{V}' = \mathcal{R} \mathbf{V} = \mathcal{R}_z \mathcal{R}_y \mathcal{R}_x \mathbf{V}$$

$$\mathcal{R} = \begin{pmatrix} \cos \alpha & \sin \alpha & 0 \\ -\sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \beta & 0 \\ 0 & 1 \\ \sin \beta & 0 \end{pmatrix} \begin{pmatrix} \cos \gamma & \sin \gamma \\ -\sin \gamma & \cos \gamma \\ 0 & 0 & 1 \end{pmatrix}$$

[http://en.wikipedia.org/wiki/Euler\\_angles](http://en.wikipedia.org/wiki/Euler_angles)

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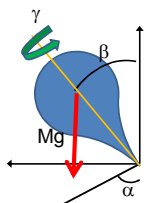
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Motion of a symmetric top under the influence of the torque of gravity:



$$L(\alpha, \beta, \gamma, \dot{\alpha}, \dot{\beta}, \dot{\gamma}) = \frac{1}{2} I_1 (\dot{\alpha}^2 \sin^2 \beta + \dot{\beta}^2) + \frac{1}{2} I_3 [\dot{\alpha} \cos \beta + \dot{\gamma}]^2 - Mgl \cos \beta$$

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$$L(\alpha, \beta, \gamma, \dot{\alpha}, \dot{\beta}, \dot{\gamma}) = \frac{1}{2} I_1 (\dot{\alpha}^2 \sin^2 \beta + \dot{\beta}^2) + \frac{1}{2} I_3 [\dot{\alpha} \cos \beta + \dot{\gamma}]^2 - Mgl \cos \beta$$

Constants of the motion :

$$p_\alpha = \frac{\partial L}{\partial \dot{\alpha}} = I_1 \dot{\alpha} \sin^2 \beta + I_3 [\dot{\alpha} \cos \beta + \dot{\gamma}] \cos \beta$$

$$p_\gamma = \frac{\partial L}{\partial \dot{\gamma}} = I_3 [\dot{\alpha} \cos \beta + \dot{\gamma}]$$

$$E = \frac{1}{2} I_1 \dot{\beta}^2 + \frac{p_\gamma^2}{2I_3} + V_{\text{eff}}(\beta)$$

$$L(\beta, \dot{\beta}) = \frac{1}{2} I_1 \dot{\beta}^2 + \frac{(p_\alpha - p_\gamma \cos \beta)^2}{2I_1 \sin^2 \beta} + \frac{p_\gamma^2}{2I_3} - Mgl \cos \beta$$

$$V_{\text{eff}}(\beta) = \frac{(p_\alpha - p_\gamma \cos \beta)^2}{2I_1 \sin^2 \beta} + Mgl \cos \beta$$

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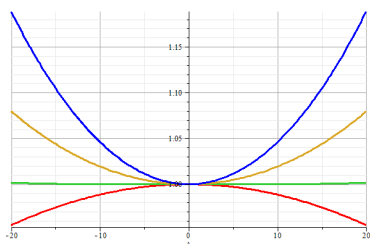
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$$E = \frac{1}{2} I_1 \dot{\beta}^2 + \frac{p_\gamma^2}{2I_3} + \frac{(p_\alpha - p_\gamma \cos \beta)^2}{2I_1 \sin^2 \beta} + Mgl \cos \beta$$

$$E' = E - \frac{p_\gamma^2}{2I_3} = \frac{1}{2} I_1 \dot{\beta}^2 + \frac{(p_\alpha - p_\gamma \cos \beta)^2}{2I_1 \sin^2 \beta} + Mgl \cos \beta$$

Stable/unstable solutions near  $\beta=0$



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Suppose  $p_\alpha = p_\gamma$  and  $\beta \approx 0$

$$E' = E - \frac{p_\gamma^2}{2I_3} = \frac{1}{2}I_1\dot{\beta}^2 + \frac{(p_\alpha - p_\gamma \cos \beta)^2}{2I_1 \sin^2 \beta} + Mgl \cos \beta$$

$$E' \approx \frac{1}{2}I_1\dot{\beta}^2 + \frac{p_\gamma^2(1 - \frac{1}{2}\beta^2)^2}{2I_1 \beta^2} + Mgl(1 - \frac{1}{2}\beta^2)$$

$$\approx \frac{1}{2}I_1\dot{\beta}^2 + \left(\frac{p_\gamma^2}{8I_1} - \frac{Mgl}{2}\right)\beta^2 + Mgl$$

$\Rightarrow$  Stable solution if

$$p_\gamma \geq \sqrt{4MglI_1}$$

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More general case:

$$E' = E - \frac{p_\gamma^2}{2I_3} = \frac{1}{2}I_1\dot{\beta}^2 + \frac{(p_\alpha - p_\gamma \cos \beta)^2}{2I_1 \sin^2 \beta} + Mgl \cos \beta$$

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Constants of the motion :

$$p_\gamma = \frac{\partial L}{\partial \dot{\gamma}} = I_3[\dot{\alpha} \cos \beta + \dot{\gamma}]$$

$$p_\alpha = \frac{\partial L}{\partial \dot{\alpha}} = I_1\dot{\alpha} \sin^2 \beta + I_3[\dot{\alpha} \cos \beta + \dot{\gamma}] \cos \beta$$

$$= I_1\dot{\alpha} \sin^2 \beta + p_\gamma \cos \beta$$

$$E' = E - \frac{p_\gamma^2}{2I_3} = \frac{1}{2}I_1\dot{\beta}^2 + \frac{(p_\alpha - p_\gamma \cos \beta)^2}{2I_1 \sin^2 \beta} + Mgl \cos \beta$$

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