PHY 711 Classical Mechanics and Mathematical Methods
11-11:50 AM MWF Olin 107

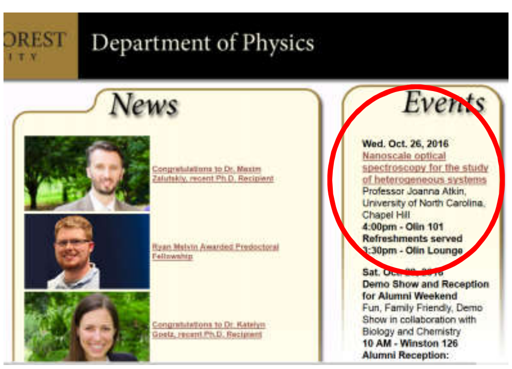
Plan for Lecture 23:

Chap. 8 in F & W: Summary of two-dimensional membrane analysis

Chap. 9 in F & W: Introduction to hydrodynamics

1. **Motivation for topic**
2. **Newton's laws for fluids**
3. **Conservation relations**

20	Mon, 10/17/2016	Chap. 5	Mechanics of rigid bodies	Exam due	
21	Wed, 10/19/2016	Chap. 5	Mechanics of rigid bodies	#16	10/24/2016
	Fri, 10/21/2016		Fall break -- no class		
22	Mon, 10/24/2016	Chap. 8	Mechanics of Elastic Membranes	#17	10/28/2016
23	Wed, 10/26/2016	Chap. 9	Introduction to hydrodynamics		
24	Fri, 10/28/2016				
25	Mon, 10/31/2016				
26	Wed, 11/02/2016				
27	Fri, 11/04/2016				
28	Mon, 11/07/2016				
29	Wed, 11/09/2016				
30	Fri, 11/11/2016				
31	Mon, 11/14/2016				
32	Wed, 11/16/2016				
33	Fri, 11/18/2016				
34	Mon, 11/21/2016				
	Wed, 11/23/2016		Thanksgiving Holiday -- no class		
	Fri, 11/25/2016		Thanksgiving Holiday -- no class		
	Wed, 12/07/2016		Presentations I		
	Fri, 12/09/2016		Presentations II		



WFU Physics and Chemistry Joint Colloquium

TITLE: Nanoscale optical spectroscopy for the study of heterogeneous systems

SPEAKER: Professor Joanna Atkin,

Department of Chemistry
University of North Carolina
Chapel Hill, North Carolina

TIME: Wednesday October 26, 2016 at 4:00 PM

PLACE: Room 101 Olin Physical Laboratory

Refreshments will be served at 3:30 PM in the Olin Lounge. All interested persons are cordially invited to attend.

ABSTRACT

Many materials classes show some of mesoscale spatial heterogeneity that is important for functional properties, ranging from biological structure-function relations to domain formation in strongly correlated electron materials. The use of ultrasharp scanning probe tips as optical antennas provides a means to overcome the diffraction limit and extend optical spectroscopy to the nanoscale, in order to investigate the causes and effects of these spatial heterogeneities. This approach is compatible with a broad range of optical modalities, including ultrafast and nonlinear implementations, enabling the study of sub-ensemble and sub-domain behavior. I will discuss several of our recent results in scattering scanning near-field optical microscopy (s-SNOM) that have demonstrated an unprecedented degree of specificity, sensitivity, and selectivity in probing nanoscale systems. In particular, with infrared and Raman vibrational s-SNOM spectroscopy, we can directly access information about conductivity variations in semiconducting nanowires and low-dimensional materials, that is important for designing and optimizing their use in devices.

Review --

Two-dimensional wave equation:

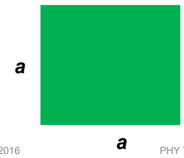
$$\frac{\partial^2 u}{\partial t^2} - c^2\nabla^2 u = 0 \qquad \text{where } c^2 = \frac{\tau}{\sigma}$$

Standing wave solutions:

$$u(x,y,t) = \Re\left(e^{-i\omega t}\rho(x,y)\right)$$

$$\left(\nabla^2 + k^2\right)\rho(x,y) = 0 \qquad \text{where } k = \frac{\omega}{c}$$

Consider a square boundary:



Free boundary conditions:

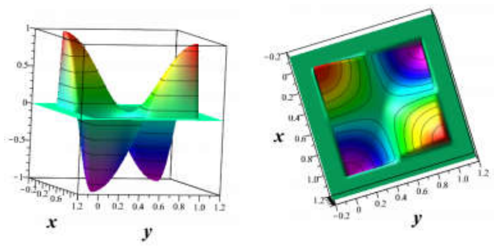
$$\frac{\partial\rho(0,y)}{\partial x} = \frac{\partial\rho(a,y)}{\partial x} = \frac{\partial\rho(x,a)}{\partial y} = \frac{\partial\rho(x,0)}{\partial y} = 0$$

$$\Rightarrow \rho_{mn}(x,y) = A\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$

$$k_{mn}^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \qquad \omega_{mn} = ck_{mn}$$

For $n = m = 1$:

$$\rho_{11}(x,y) = A\cos\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi y}{b}\right)$$



Hydrodynamic analysis
Motivation

1. Natural progression from strings, membranes, fluids; description of 1, 2, and 3 dimensional continua
2. Interesting and technologically important phenomena associated with fluids

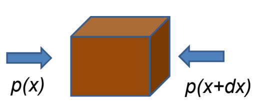
Plan

1. Newton's laws for fluids
2. Continuity equation
3. Stress tensor
4. Energy relations
5. Bernoulli's theorem
6. Various examples
7. Sound waves

Newton's equations for fluids
Use Euler formulation; following "particles" of fluid

$$\text{Variables:}\quad \text{Density}\quad \rho(x,y,z,t)$$
$$\text{Pressure}\quad p(x,y,z,t)$$
$$\text{Velocity}\quad \mathbf{v}(x,y,z,t)$$

$$m\mathbf{a} = \mathbf{F}$$
$$m \rightarrow \rho dV$$
$$\mathbf{a} \rightarrow \frac{d\mathbf{v}}{dt}$$
$$\mathbf{F} \rightarrow \mathbf{F}_{applied} + \mathbf{F}_{pressure}$$



$$F_{pressure}\Big|_x = \left(-\,p(x+dx,y,z)+p(x,y,z)\right)dydz$$
$$= \frac{\left(-\,p(x+dx,y,z)+p(x,y,z)\right)}{dx}dxdydz$$
$$= -\frac{\partial p}{\partial x}dV$$

Newton's equations for fluids -- continued

$$m\mathbf{a} = \mathbf{F}_{applied} + \mathbf{F}_{pressure}$$

$$\rho dV \frac{d\mathbf{v}}{dt} = \mathbf{f}_{applied}\,\rho dV - (\nabla p)dV$$

$$\rho \frac{d\mathbf{v}}{dt} = \rho \mathbf{f}_{applied} - \nabla p$$

Detailed analysis of acceleration term:

$$\mathbf{v} = \mathbf{v}(x, y, z, t)$$

$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial x}\frac{dx}{dt} + \frac{\partial \mathbf{v}}{\partial y}\frac{dy}{dt} + \frac{\partial \mathbf{v}}{\partial z}\frac{dz}{dt} + \frac{\partial \mathbf{v}}{\partial t}$$

$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial x}v_x + \frac{\partial \mathbf{v}}{\partial y}v_y + \frac{\partial \mathbf{v}}{\partial z}v_z + \frac{\partial \mathbf{v}}{\partial t}$$

$$\frac{d\mathbf{v}}{dt} = (\mathbf{v}\cdot\nabla)\mathbf{v} + \frac{\partial \mathbf{v}}{\partial t}$$

Note that:

$$\frac{\partial \mathbf{v}}{\partial x}v_x + \frac{\partial \mathbf{v}}{\partial y}v_y + \frac{\partial \mathbf{v}}{\partial z}v_z = \nabla\left(\frac{1}{2}\mathbf{v}\cdot\mathbf{v}\right) - \mathbf{v}\times(\nabla\times\mathbf{v})$$

Newton's equations for fluids -- continued

$$\rho \frac{d\mathbf{v}}{dt} = \rho \mathbf{f}_{applied} - \nabla p$$

$$\rho\left(\nabla\left(\frac{1}{2}\mathbf{v}\cdot\mathbf{v}\right) - \mathbf{v}\times(\nabla\times\mathbf{v}) + \frac{\partial \mathbf{v}}{\partial t}\right) = \rho \mathbf{f}_{applied} - \nabla p$$

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) - \mathbf{v}\times(\nabla\times\mathbf{v}) = \mathbf{f}_{applied} - \frac{\nabla p}{\rho}$$

Solution of Euler's equation for fluids

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) - \mathbf{v}\times\left(\nabla\times\mathbf{v}\right) = \mathbf{f}_{applied} - \frac{\nabla p}{\rho}$$

Consider the following restrictions:

1. $(\nabla\times\mathbf{v}) = 0$ "irrotational flow"

   $\Rightarrow \mathbf{v} = -\nabla\Phi$

2. $\mathbf{f}_{applied} = -\nabla U$ conservative applied force

3. $\rho = \text{(constant)}$ incompressible fluid

$$\frac{\partial(-\nabla\Phi)}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) = -\nabla U - \frac{\nabla p}{\rho}$$

$$\Rightarrow \nabla\left(\frac{p}{\rho} + U + \tfrac{1}{2}v^2 - \frac{\partial\Phi}{\partial t}\right) = 0$$

Bernoulli's integral of Euler's equation

$$\nabla\left(\frac{p}{\rho} + U + \tfrac{1}{2}v^2 - \frac{\partial\Phi}{\partial t}\right) = 0$$

Integrating over space:

$$\frac{p}{\rho} + U + \tfrac{1}{2}v^2 - \frac{\partial\Phi}{\partial t} = C(t)$$

where $\mathbf{v} = -\nabla\Phi(\mathbf{r},t) = -\nabla\left(\Phi(\mathbf{r},t) + C'(t)\right)$

$$\Rightarrow \frac{p}{\rho} + U + \tfrac{1}{2}v^2 - \frac{\partial\Phi}{\partial t} = 0 \qquad \text{Bernoulli's theorem}$$

Examples of Bernoulli's theorem

$$\frac{p}{\rho} + U + \tfrac{1}{2}v^2 - \frac{\partial\Phi}{\partial t} = 0$$

Modified form; assuming $\dfrac{\partial\Phi}{\partial t} = 0$

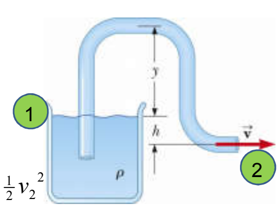
$$\frac{p}{\rho} + U + \tfrac{1}{2}v^2 = \text{constant}$$

$p_1 = p_2 = p_{atm}$

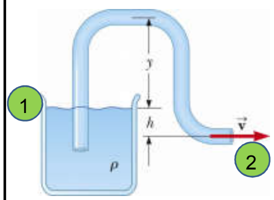
$U_1 - U_2 = gh$

$v_1 \approx 0$

$$\frac{p_1}{\rho} + U_1 + \tfrac{1}{2}{v_1}^2 = \frac{p_2}{\rho} + U_2 + \tfrac{1}{2}{v_2}^2$$



Examples of Bernoulli's theorem -- continued



$$p_1 = p_2 = p_{atm}$$
$$U_1 - U_2 = gh$$
$$v_1 \approx 0$$
$$\frac{p_1}{\rho} + U_1 + \tfrac{1}{2}v_1^2 = \frac{p_2}{\rho} + U_2 + \tfrac{1}{2}v_2^2$$

$$v_2 \approx \sqrt{2gh}$$

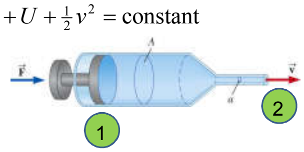
Examples of Bernoulli's theorem -- continued

$$\frac{p}{\rho} + U + \tfrac{1}{2}v^2 = \text{constant}$$

$$p_1 = \frac{F}{A} + p_{atm} \qquad p_2 = p_{atm}$$
$$U_1 = U_2$$
$$v_1 A = v_2 a \qquad \text{continuity equation}$$
$$\frac{p_1}{\rho} + U_1 + \tfrac{1}{2}v_1^2 = \frac{p_2}{\rho} + U_2 + \tfrac{1}{2}v_2^2$$

Examples of Bernoulli's theorem -- continued

$$\frac{p}{\rho} + U + \tfrac{1}{2}v^2 = \text{constant}$$



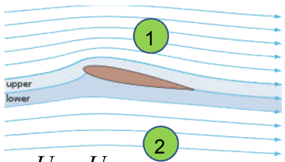
$$\frac{2F}{A} = v_2^2\left(1 - \left(\frac{a}{A}\right)^2\right)$$
$$v_2 = \sqrt{\frac{2F/A}{1 - \left(\frac{a}{A}\right)^2}}$$

Examples of Bernoulli's theorem – continued

Approximate explanation of airplane lift

Cross section view of airplane wing

http://en.wikipedia.org/wiki/Lift_%28force%29



$$U_1 \approx U_2$$

$$\frac{p_1}{\rho} + U_1 + \tfrac{1}{2}v_1^2 = \frac{p_2}{\rho} + U_2 + \tfrac{1}{2}v_2^2$$

$$p_2 - p_1 = \tfrac{1}{2}\rho\left(v_1^2 - v_2^2\right)$$

Continuity equation connecting fluid density and velocity:

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0$$

$$\frac{\partial \rho}{\partial t} + \rho(\nabla\cdot\mathbf{v}) + (\nabla\rho)\cdot\mathbf{v} = 0$$

$$\text{Consider: } \frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + (\nabla\rho)\cdot\mathbf{v}$$

$$\Rightarrow \frac{d\rho}{dt} + \rho(\nabla\cdot\mathbf{v}) = 0$$

alternative form of continuity equation

Some details on the velocity potential

Continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0$$

$$\frac{\partial \rho}{\partial t} + \rho(\nabla\cdot\mathbf{v}) + (\nabla\rho)\cdot\mathbf{v} = 0$$

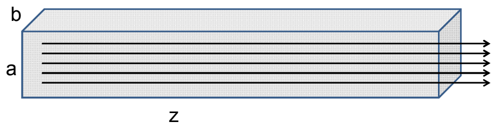
For incompressible fluid: $\rho = (\text{constant})$

$$\Rightarrow \nabla\cdot\mathbf{v} = 0$$

Irrotational flow: $\nabla\times\mathbf{v} = 0 \qquad \Rightarrow \mathbf{v} = -\nabla\Phi$

$$\Rightarrow \nabla^2\Phi = 0$$

Example – uniform flow



$$\nabla^2\Phi = 0$$

$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2\Phi}{\partial z^2}=0$$

Possible solution :

$$\Phi = -v_o z$$

$$\mathbf{v} = -\nabla\Phi = v_o\hat{\mathbf{z}}$$

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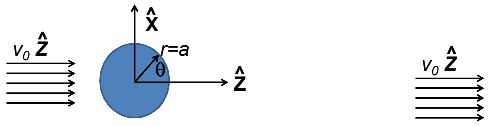
Example – flow around a long cylinder (oriented in the ***Y*** direction)



$$\nabla^2\Phi = 0$$

$$\left.\frac{\partial\Phi}{\partial r}\right|_{r=a} = 0$$

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Laplace equation in cylindrical coordinates

(r,θ, defined in x-z plane; y representing cylinder axis)

$$\nabla^2\Phi = 0 = \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial\Phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2}+\frac{\partial^2\Phi}{\partial y^2}$$

In our case, there is no motion in the y dimension

$$\Rightarrow \Phi(r,\theta,y) = \Phi(r,\theta)$$

From boundary condition : $v_z(r\to\infty) = v_0$

$$\frac{\partial\Phi}{\partial z}(r\to\infty) = -v_0 \qquad \Rightarrow \Phi(r\to\infty,\theta) = -v_0 r\cos\theta$$

Note that : $\dfrac{\partial^2\cos\theta}{\partial\theta^2} = -\cos\theta$

Guess form : $\Phi(r,\theta) = f(r)\cos\theta$

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Necessary equation for radial function

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial f}{\partial r}-\frac{1}{r^2}f=0$$

$$f(r)=Ar+\frac{B}{r} \qquad \text{where } A, B \text{ are constants}$$

Boundary condition on cylinder surface:

$$\left.\frac{\partial \Phi}{\partial r}\right|_{r=a}=0$$

$$\frac{df}{dr}(r=a)=0=A-\frac{B}{a^2}$$

$$\Rightarrow B=Aa^2$$

Boundary condition at ∞: $\Rightarrow A=-v_0$

$$\Phi(r,\theta)=-v_0\left(r+\frac{a^2}{r}\right)\cos\theta$$

$$v_r=-\frac{\partial \Phi}{\partial r}=v_0\left(1-\frac{a^2}{r^2}\right)\cos\theta$$

$$v_\theta=-\frac{1}{r}\frac{\partial \Phi}{\partial \theta}=v_0\left(1+\frac{a^2}{r^2}\right)\sin\theta$$

For 3-dimensional system, consider a spherical obstruction

Laplacian in spherical polar coordinates:

$$\nabla^2\Phi=0=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \Phi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \Phi}{\partial \theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial \varphi^2}$$

to be continued …