

**PHY 711 Classical Mechanics and
Mathematical Methods**
11-11:50 AM MWF Olin 107

Plan for Lecture 24:

Introduction to hydrodynamics

(Chap. 9 in F & W)

1. Incompressible fluids

2. Isentropic fluids

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20 Mon, 10/17/2016	Chap. 5	Mechanics of rigid bodies	Exam due	
21 Wed, 10/19/2016	Chap. 5	Mechanics of rigid bodies	#16	10/24/2016
Fri, 10/21/2016		Fall break -- no class		
22 Mon, 10/24/2016	Chap. 8	Mechanics of Elastic Membranes	#17	10/28/2016
23 Wed, 10/26/2016	Chap. 9	Introduction to hydrodynamics	#18	10/31/2016
24 Fri, 10/28/2016	Chap. 9	Introduction to hydrodynamics	#18	10/31/2016
25 Mon, 10/31/2016				
26 Wed, 11/02/2016				
27 Fri, 11/04/2016				
28 Mon, 11/07/2016				
29 Wed, 11/09/2016				
30 Fri, 11/11/2016				
31 Mon, 11/14/2016				
32 Wed, 11/16/2016				
33 Fri, 11/18/2016				
34 Mon, 11/21/2016				
Wed, 11/23/2016		Thanksgiving Holiday -- no class		
Fri, 11/25/2016		Thanksgiving Holiday -- no class		
Wed, 12/07/2016		Presentations I		
Fri, 12/09/2016		Presentations II		

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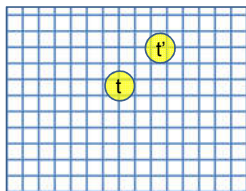
Newton's equations for fluids

Use **Euler** formulation; properties described in terms of stationary spatial grid

Variables: Density $\rho(x,y,z,t)$

Pressure $p(x,y,z,t)$

Velocity $\mathbf{v}(x,y,z,t)$



Particle at t : $\mathbf{r}, t$

Particle at t' : $\mathbf{r} + \mathbf{v} \delta t, t'$

$$t' = t + \delta t$$

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Euler analysis -- continued

Particle at t : $\mathbf{r}, t$

Particle at t' : $\mathbf{r} + \mathbf{v}\delta t, t'$ where $\delta t = t' - t$

For $f(\mathbf{r}, t)$:

$$\frac{df}{dt} = \lim_{\delta t \rightarrow 0} \left(\frac{f(\mathbf{r}, t') - f(\mathbf{r}, t)}{\delta t} + \frac{f(\mathbf{r} + \mathbf{v}\delta t, t) - f(\mathbf{r}, t)}{\delta t} \right)$$

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + (\mathbf{v} \cdot \nabla) f$$

Example: $(\mathbf{v} \cdot \nabla) \mathbf{v} = \nabla \left(\frac{1}{2} v^2 \right) - \mathbf{v} \times (\nabla \times \mathbf{v})$

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Some details on the velocity potential

Continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial \rho}{\partial t} + \rho(\nabla \cdot \mathbf{v}) + (\nabla \rho) \cdot \mathbf{v} = 0$$

For incompressible fluid: $\rho = (\text{constant})$

$$\Rightarrow \nabla \cdot \mathbf{v} = 0$$

Irrotational flow: $\nabla \times \mathbf{v} = 0 \quad \Rightarrow \mathbf{v} = -\nabla \Phi$

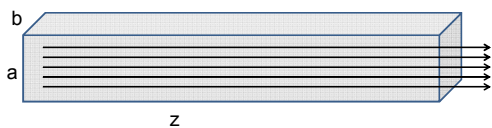
$$\Rightarrow \nabla^2 \Phi = 0$$

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Example -- uniform flow



$$\nabla^2 \Phi = 0$$

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

Possible solution:

$$\Phi = -v_o z$$

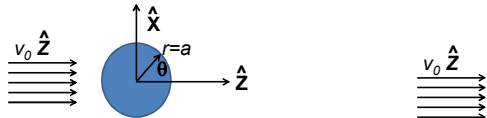
$$\mathbf{v} = -\nabla \Phi = v_o \hat{\mathbf{z}}$$

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Example – flow around a long cylinder (oriented in the Y direction)



$$\nabla^2 \Phi = 0$$

$$\left. \frac{\partial \Phi}{\partial r} \right|_{r=a} = 0$$

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Laplace equation in cylindrical coordinates

(r, θ , defined in x - z plane; y representing cylinder axis)

$$\nabla^2 \Phi = 0 = \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial \Phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \theta^2} + \frac{\partial^2 \Phi}{\partial y^2}$$

In our case, there is no motion in the y dimension

$$\Rightarrow \Phi(r, \theta, y) = \Phi(r, \theta)$$

From boundary condition : $v_z(r \rightarrow \infty) = v_0$

$$\frac{\partial \Phi}{\partial z}(r \rightarrow \infty) = -v_0 \quad \Rightarrow \quad \Phi(r \rightarrow \infty, \theta) = -v_0 r \cos \theta$$

$$\text{Note that : } \frac{\partial^2 \cos \theta}{\partial \theta^2} = -\cos \theta$$

$$\text{Guess form : } \Phi(r, \theta) = f(r) \cos \theta$$

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Necessary equation for radial function

$$\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial f}{\partial r} - \frac{1}{r^2} f = 0$$

$$f(r) = Ar + \frac{B}{r} \quad \text{where } A, B \text{ are constants}$$

Boundary condition on cylinder surface :

$$\left. \frac{\partial \Phi}{\partial r} \right|_{r=a} = 0$$

$$\frac{df}{dr}(r=a) = 0 = A - \frac{B}{a^2}$$

$$\Rightarrow B = Aa^2$$

Boundary condition at ∞ : $\Rightarrow A = -v_0$

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$$\Phi(r, \theta) = -v_0 \left(r + \frac{a^2}{r} \right) \cos \theta$$

$$v_r = -\frac{\partial \Phi}{\partial r} = v_0 \left(1 - \frac{a^2}{r^2} \right) \cos \theta$$

$$v_\theta = -\frac{1}{r} \frac{\partial \Phi}{\partial \theta} = -v_0 \left(1 + \frac{a^2}{r^2} \right) \sin \theta$$

For 3-dimensional system, consider a spherical obstruction
Laplacian in spherical polar coordinates:

$$\nabla^2 \Phi = 0 = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \Phi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \Phi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \Phi}{\partial \phi^2}$$

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Spherical system continued:

Laplacian in spherical polar coordinates:

$$\nabla^2 \Phi = 0 = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \Phi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \Phi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \Phi}{\partial \phi^2}$$

In terms of spherical harmonic functions:

$$\left(\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right) Y_{lm}(\theta, \phi) = -l(l+1) Y_{lm}(\theta, \phi)$$

In our case:

$$Y_{10}(\theta, \phi) = \sqrt{\frac{3}{4\pi}} \cos \theta$$

$$\Phi(r, \theta, \phi) = f(r) Y_{10}(\theta, \phi)$$

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{df}{dr} \right) - \frac{l(l+1)}{r^2} f = 0 \quad \text{(Continue analysis for homework)}$$

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Solution of Euler's equation for fluids

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla \left(\frac{1}{2} v^2 \right) - \mathbf{v} \times (\nabla \times \mathbf{v}) = \mathbf{f}_{\text{applied}} - \frac{\nabla p}{\rho}$$

Consider the following restrictions:

1. $(\nabla \times \mathbf{v}) = 0$ "irrotational flow"

$$\Rightarrow \mathbf{v} = -\nabla \Phi$$

2. $\mathbf{f}_{\text{applied}} = -\nabla U$ conservative applied force

3. $\rho = (\text{constant})$ incompressible fluid

$$\frac{\partial(-\nabla \Phi)}{\partial t} + \nabla \left(\frac{1}{2} v^2 \right) = -\nabla U - \frac{\nabla p}{\rho}$$

$$\Rightarrow \nabla \left(\frac{p}{\rho} + U + \frac{1}{2} v^2 - \frac{\partial \Phi}{\partial t} \right) = 0$$

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Bernoulli's integral of Euler's equation for constant ρ

$$\nabla \left(\frac{p}{\rho} + U + \frac{1}{2} v^2 - \frac{\partial \Phi}{\partial t} \right) = 0$$

Integrating over space:

$$\frac{p}{\rho} + U + \frac{1}{2} v^2 - \frac{\partial \Phi}{\partial t} = C(t)$$

where $\mathbf{v} = -\nabla \Phi(\mathbf{r}, t) = -\nabla(\Phi(\mathbf{r}, t) + C(t))$

$$\Rightarrow \frac{p}{\rho} + U + \frac{1}{2} v^2 - \frac{\partial \Phi}{\partial t} = C_0 \quad \begin{array}{l} \text{Bernoulli's theorem} \\ \text{For incompressible fluid} \end{array}$$

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Solution of Euler's equation for fluids -- isentropic

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla \left(\frac{1}{2} v^2 \right) - \mathbf{v} \times (\nabla \times \mathbf{v}) = \mathbf{f}_{\text{applied}} - \frac{\nabla p}{\rho}$$

Consider the following restrictions:

1. $(\nabla \times \mathbf{v}) = 0$ "irrotational flow"

$$\Rightarrow \mathbf{v} = -\nabla \Phi$$

2. $\mathbf{f}_{\text{applied}} = -\nabla U$ conservative applied force

3. $\rho \neq (\text{constant})$ isentropic fluid

A little thermodynamics

First law of thermodynamics: $dE_{\text{int}} = dQ - dW$

For isentropic conditions: $dQ = 0$

$$dE_{\text{int}} = -dW = -pdV$$

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Solution of Euler's equation for fluids -- isentropic (continued)

$$dE_{\text{int}} = -dW = pdV$$

In terms of mass density: $\rho = \frac{M}{V}$

For fixed M and variable V : $d\rho = -\frac{M}{V^2} dV$

$$dV = -\frac{M}{\rho^2} d\rho$$

In terms in intensive variables: Let $E_{\text{int}} = M\varepsilon$

$$dE_{\text{int}} = Md\varepsilon = -dW = -pdV = M \frac{p}{\rho^2} d\rho$$

$$d\varepsilon = \frac{p}{\rho^2} d\rho \quad \left(\frac{\partial \varepsilon}{\partial \rho} \right)_{dQ=0} = \frac{p}{\rho^2}$$

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Solution of Euler's equation for fluids – isentropic (continued)

$$\left(\frac{\partial \varepsilon}{\partial \rho}\right)_{dQ=0} = \frac{p}{\rho^2}$$

Consider: $\nabla \varepsilon = \left(\frac{\partial \varepsilon}{\partial \rho}\right)_{dQ=0} \nabla \rho = \frac{p}{\rho^2} \nabla \rho$

Rearranging: $\nabla \left(\varepsilon + \frac{p}{\rho} \right) = \frac{\nabla p}{\rho}$

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Solution of Euler's equation for fluids – isentropic (continued)

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla \left(\frac{1}{2} v^2 \right) - \mathbf{v} \times (\nabla \times \mathbf{v}) = \mathbf{f}_{\text{applied}} - \frac{\nabla p}{\rho}$$

$$\frac{\nabla p}{\rho} = \nabla \left(\varepsilon + \frac{p}{\rho} \right)$$

$$\nabla \times \mathbf{v} = 0 \quad \mathbf{v} = -\nabla \Phi \quad \mathbf{f}_{\text{applied}} = -\nabla U$$

$$\frac{\partial(-\nabla \Phi)}{\partial t} + \nabla \left(\frac{1}{2} v^2 \right) = -\nabla U - \nabla \left(\varepsilon + \frac{p}{\rho} \right)$$

$$\Rightarrow \nabla \left(\varepsilon + \frac{p}{\rho} + U + \frac{1}{2} v^2 - \frac{\partial \Phi}{\partial t} \right) = 0$$

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Summary of Bernoulli's results

For incompressible fluid

$$\nabla \left(\frac{p}{\rho} + U + \frac{1}{2} v^2 - \frac{\partial \Phi}{\partial t} \right) = 0$$

For isentropic fluid with internal energy density ε

$$\nabla \left(\varepsilon + \frac{p}{\rho} + U + \frac{1}{2} v^2 - \frac{\partial \Phi}{\partial t} \right) = 0$$

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Application of fluid equations to the case of air in equilibrium plus small perturbation

Newton - Euler equation of motion :

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = \mathbf{f}_{\text{applied}} - \frac{\nabla p}{\rho}$$

Continuity equation : $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$

Near equilibrium :

$$\rho = \rho_0 + \delta \rho$$

$$p = p_0 + \delta p$$

$$\mathbf{v} = 0 + \delta \mathbf{v}$$

$$\mathbf{f}_{\text{applied}} = 0$$

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Equations to lowest order in perturbation :

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = \mathbf{f}_{\text{applied}} - \frac{\nabla p}{\rho} \quad \Rightarrow \quad \frac{\partial \delta \mathbf{v}}{\partial t} = - \frac{\nabla \delta p}{\rho_0}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \quad \Rightarrow \quad \frac{\partial \delta \rho}{\partial t} + \rho_0 \nabla \cdot \delta \mathbf{v} = 0$$

In terms of the velocity potential :

$$\delta \mathbf{v} = -\nabla \Phi$$

$$\frac{\partial \delta \mathbf{v}}{\partial t} = - \frac{\nabla \delta p}{\rho_0} \quad \Rightarrow \quad \nabla \left(- \frac{\partial \Phi}{\partial t} + \frac{\delta p}{\rho_0} \right) = 0$$

$$\frac{\partial \delta \rho}{\partial t} + \rho_0 \nabla \cdot \delta \mathbf{v} = 0 \quad \Rightarrow \quad \frac{\partial \delta \rho}{\partial t} - \rho_0 \nabla^2 \Phi = 0$$

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Expressing pressure in terms of the density :

$$p = p(s, \rho) = p_0 + \delta p \quad \text{where } s \text{ denotes the (constant) entropy}$$

$$p_0 = p(s, \rho_0)$$

$$\delta p = \left(\frac{\partial p}{\partial \rho} \right)_s \delta \rho \equiv c^2 \delta \rho$$

$$\nabla \left(- \frac{\partial \Phi}{\partial t} + \frac{\delta p}{\rho_0} \right) = 0 \quad \Rightarrow \quad - \frac{\partial \Phi}{\partial t} + c^2 \frac{\delta \rho}{\rho_0} = (\text{constant})$$

$$\Rightarrow - \frac{\partial^2 \Phi}{\partial t^2} + \frac{c^2}{\rho_0} \frac{\partial \delta \rho}{\partial t} = 0$$

$$\frac{\partial \delta \rho}{\partial t} - \rho_0 \nabla^2 \Phi = 0 \quad \Rightarrow \quad \frac{\partial^2 \Phi}{\partial t^2} - c^2 \nabla^2 \Phi = 0$$

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Wave equation for air :

$$\frac{\partial^2 \Phi}{\partial t^2} - c^2 \nabla^2 \Phi = 0$$

Here, $c^2 = \left(\frac{\partial p}{\partial \rho} \right)_s$

$$\mathbf{v} = -\nabla \Phi$$

Boundary values:

Impenetrable surface with normal $\hat{\mathbf{n}}$ moving at velocity $\mathbf{V}$:

$$\hat{\mathbf{n}} \cdot \mathbf{V} = \hat{\mathbf{n}} \cdot \delta \mathbf{v} = -\hat{\mathbf{n}} \cdot \nabla \Phi$$

Free surface:

$$\delta p = 0 \quad \Rightarrow \quad \rho_0 \frac{\partial \Phi}{\partial t} = 0$$

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Analysis of wave velocity in an ideal gas:

$$c^2 = \left(\frac{\partial p}{\partial \rho} \right)_s$$

Equation of state for ideal gas :

$$pV = NkT \quad N = \frac{M}{M_0}$$

$$p = \frac{M}{V} \frac{k}{M_0} T = \rho \frac{k}{M_0} T$$

$$k = 1.38 \times 10^{-23} \text{ J / K}$$

M_0 = average mass of each molecule

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Internal energy for ideal gas :

$$E = \frac{f}{2} NkT = M\varepsilon \quad \varepsilon = \frac{f}{2} \frac{k}{M_0} T = \frac{f}{2} \frac{p}{\rho}$$

In terms of specific heat ratio : $\gamma \equiv \frac{C_p}{C_v}$

$$dE = dQ - dW$$

$$C_v = \left(\frac{dQ}{dT} \right)_v = \left(\frac{\partial E}{\partial T} \right)_v = \frac{f}{2} \frac{Mk}{M_0}$$

$$C_p = \left(\frac{dQ}{dT} \right)_p = \left(\frac{\partial E}{\partial T} \right)_p + p \left(\frac{\partial V}{\partial T} \right)_p = \frac{f}{2} \frac{Mk}{M_0} + \frac{Mk}{M_0}$$

$$\gamma = \frac{C_p}{C_v} = \frac{\frac{f}{2} + 1}{\frac{f}{2}} \quad \Rightarrow \quad \frac{f}{2} = \frac{1}{\gamma - 1}$$

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Internal energy for ideal gas :

$$E = \frac{1}{\gamma-1} NkT = M\varepsilon \quad \varepsilon = \frac{1}{\gamma-1} \frac{k}{M_0} T = \frac{1}{\gamma-1} \frac{p}{\rho}$$

Internal energy for ideal gas under isentropic conditions :

$$d\varepsilon = -\frac{p}{M} dV = \frac{p}{\rho^2} d\rho$$

$$\left(\frac{\partial \varepsilon}{\partial \rho}\right)_s = \frac{p}{\rho^2} = \frac{\partial}{\partial \rho} \left(\frac{1}{\gamma-1} \frac{p}{\rho} \right)_s = \left(\frac{\partial p}{\partial \rho}\right)_s \frac{1}{(\gamma-1)\rho} - \frac{p}{(\gamma-1)\rho^2}$$

$$\Rightarrow \left(\frac{\partial p}{\partial \rho}\right)_s = \frac{p\gamma}{\rho}$$

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Alternative derivation:

Isentropic or adiabatic equation of state :

$$\frac{dp}{p} = \gamma \frac{d\rho}{\rho} \quad \Rightarrow \frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^\gamma$$

$$\left(\frac{\partial p}{\partial \rho}\right)_s = \frac{p\gamma}{\rho}$$

Linearized speed of sound

$$c_0^2 = \left(\frac{\partial p}{\partial \rho}\right)_{s, p_0, \rho_0} = \frac{p_0 \gamma}{\rho_0}$$

$$c_0^2 \approx \frac{1.5 \cdot 1.013 \times 10^5 \text{ Pa}}{1.3 \text{ kg/m}^3} \quad c_0 \approx 340 \text{ m/s}$$

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Density dependence of speed of sound for ideal gas :

$$c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s = \frac{p\gamma}{\rho}$$

$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^\gamma$$

$$c^2 = \frac{p_0 \gamma}{\rho_0} \frac{p/p_0}{\rho/\rho_0} = c_0^2 \left(\frac{\rho}{\rho_0}\right)^{\gamma-1}$$

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