PHY 711 Classical Mechanics and Mathematical Methods
11-11:50 AM MWF Olin 107

Plan for Lecture 25: Chap. 9 of F&W

Wave equation for sound in linear approximation

1. Wave equations for sound
2. Speed of sound
3. Standing wave solutions

18	Wed, 10/12/2016	Chap. 7	Fourier series and transforms	Take-home exam	
19	Fri, 10/14/2016	App. A	Laplace transforms and contour integrals	Take-home exam	
20	Mon, 10/17/2016	Chap. 5	Mechanics of rigid bodies	Exam due	
21	Wed, 10/19/2016	Chap. 5	Mechanics of rigid bodies	#16	10/24/2016
	Fri, 10/21/2016		Fall break -- no class		
22	Mon, 10/24/2016	Chap. 8	Mechanics of Elastic Membranes	#17	10/28/2016
23	Wed, 10/26/2016	Chap. 9	Introduction to hydrodynamics		
24	Fri, 10/28/2016	Chap. 9	Introduction to hydrodynamics	#18	10/31/2016
25	Mon, 10/31/2016	Chap. 9	Sound waves	#19	11/02/2016
26	Wed, 11/02/2016				
27	Fri, 11/04/2016				
28	Mon, 11/07/2016				
29	Wed, 11/09/2016				
30	Fri, 11/11/2016				
31	Mon, 11/14/2016				
32	Wed, 11/16/2016				
33	Fri, 11/18/2016				
34	Mon, 11/21/2016				
	Wed, 11/23/2016		Thanksgiving Holiday -- no class		
	Fri, 11/25/2016		Thanksgiving Holiday -- no class		
	Wed, 12/07/2016		Presentations I		
	Fri, 12/09/2016		Presentations II		

Application of fluid equations to the case of air in equilibrium plus small perturbation

Newton - Euler equation of motion :

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{f}_{applied} - \frac{\nabla p}{\rho}$$

Continuity equation : $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$

Near equilibrium :

$$\rho = \rho_0 + \delta\rho$$

$$p = p_0 + \delta p$$

$$\mathbf{v} = 0 + \delta\mathbf{v}$$

$$\mathbf{f}_{applied} = 0$$

Equations to lowest order in perturbation :

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{f}_{applied} - \frac{\nabla p}{\rho} \quad \Rightarrow \quad \frac{\partial \delta\mathbf{v}}{\partial t} = -\frac{\nabla \delta p}{\rho_0}$$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0 \quad \Rightarrow \quad \frac{\partial \delta\rho}{\partial t} + \rho_0 \nabla\cdot\delta\mathbf{v} = 0$$

In terms of the velocity potential :

$$\delta\mathbf{v} = -\nabla\Phi$$

$$\frac{\partial \delta\mathbf{v}}{\partial t} = -\frac{\nabla \delta p}{\rho_0} \quad \Rightarrow \nabla\left(-\frac{\partial\Phi}{\partial t} + \frac{\delta p}{\rho_0}\right) = 0$$

$$\frac{\partial \delta\rho}{\partial t} + \rho_0 \nabla\cdot\delta\mathbf{v} = 0 \quad \Rightarrow \frac{\partial \delta\rho}{\partial t} - \rho_0\nabla^2\Phi = 0$$

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Expressing pressure in terms of the density :

$p = p(s,\rho) = p_0 + \delta p$ where s denotes the (constant) entropy

$p_0 = p(s,\rho_0)$

$$\delta p = \left(\frac{\partial p}{\partial \rho}\right)_s \delta\rho \equiv c^2\delta\rho$$

$$\nabla\left(-\frac{\partial\Phi}{\partial t} + \frac{\delta p}{\rho_0}\right) = 0 \quad \Rightarrow -\frac{\partial\Phi}{\partial t} + c^2\frac{\delta\rho}{\rho_0} = (\text{constant})$$

$$\Rightarrow -\frac{\partial^2\Phi}{\partial t^2} + \frac{c^2}{\rho_0}\frac{\partial\delta\rho}{\partial t} = 0$$

$$\frac{\partial \delta\rho}{\partial t} - \rho_0\nabla^2\Phi = 0 \quad \Rightarrow \frac{\partial^2\Phi}{\partial t^2} - c^2\nabla^2\Phi = 0$$

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Wave equation for air :

$$\frac{\partial^2\Phi}{\partial t^2} - c^2\nabla^2\Phi = 0$$

Here, $c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s$

$\mathbf{v} = -\nabla\Phi$

Note that, we also have :

$$\frac{\partial^2\delta\rho}{\partial t^2} - c^2\nabla^2\delta\rho = 0$$

$$\frac{\partial^2\delta p}{\partial t^2} - c^2\nabla^2\delta p = 0$$

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Analysis of wave velocity in an ideal gas:

$$c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s$$ ⟸ adiabatic conditions

Equation of state for ideal gas:

$$pV = NkT \qquad N = \frac{M}{M_0}$$

$$p = \frac{M}{V}\frac{k}{M_0}T = \rho\frac{k}{M_0}T$$

$$k = 1.38\times 10^{-23}\,J/K$$

$M_0 =$ average mass of each molecule

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Internal energy for ideal gas :

$$E = \frac{f}{2}NkT = M\varepsilon \qquad \varepsilon = \frac{f}{2}\frac{k}{M_0}T = \frac{f}{2}\frac{p}{\rho}$$

In terms of specific heat ratio : $\gamma \equiv \frac{C_p}{C_V}$

$$dE = dQ - dW$$

$$C_V = \left(\frac{dQ}{dT}\right)_V = \left(\frac{\partial E}{\partial T}\right)_V = \frac{f}{2}\frac{Mk}{M_0}$$

$$C_p = \left(\frac{dQ}{dT}\right)_p = \left(\frac{\partial E}{\partial T}\right)_p + p\left(\frac{\partial V}{\partial T}\right)_p = \frac{f}{2}\frac{Mk}{M_0} + \frac{Mk}{M_0}$$

$$\gamma = \frac{C_p}{C_V} = \frac{\frac{f}{2}+1}{\frac{f}{2}} \qquad \Rightarrow \frac{f}{2} = \frac{1}{\gamma - 1}$$

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Internal energy for ideal gas :

$$E = \frac{1}{\gamma-1}NkT = M\varepsilon \qquad \varepsilon = \frac{1}{\gamma-1}\frac{k}{M_0}T = \frac{1}{\gamma-1}\frac{p}{\rho}$$

Internal energy for ideal gas under isentropic conditions :

$$d\varepsilon = -\frac{p}{M}dV = \frac{p}{\rho^2}d\rho$$

$$\left(\frac{\partial \varepsilon}{\partial \rho}\right)_s = \frac{p}{\rho^2} = \frac{\partial}{\partial \rho}\left(\frac{1}{\gamma-1}\frac{p}{\rho}\right)_s = \left(\frac{\partial p}{\partial \rho}\right)_s \frac{1}{(\gamma-1)\rho} - \frac{p}{(\gamma-1)\rho^2}$$

$$\Rightarrow \left(\frac{\partial p}{\partial \rho}\right)_s = \frac{p\gamma}{\rho}$$

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Alternative derivation:

Isentropic or adiabatic equation of state:

$$\frac{dp}{p} = \gamma \frac{d\rho}{\rho} \qquad \Rightarrow \frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma}$$

$$\left(\frac{\partial p}{\partial \rho}\right)_s = \frac{p\gamma}{\rho}$$

Linearized speed of sound

$$c_0^{\,2} = \left(\frac{\partial p}{\partial \rho}\right)_{s,p_0,\rho_0} = \frac{p_0 \gamma}{\rho_0}$$

$$c_0^{\,2} \approx \frac{1.5 \cdot 1.0 \times 10^5\, Pa}{1.3 kg/m^3} \qquad c_0 \approx 340 \text{ m/s}$$

Density dependence of speed of sound for ideal gas:

$$c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s = \frac{p\gamma}{\rho}$$

$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma}$$

$$c^2 = \frac{p_0\gamma}{\rho_0}\frac{p/p_0}{\rho/\rho_0} = c_0^2\left(\frac{\rho}{\rho_0}\right)^{\gamma-1}$$

where $\quad c_0^2 = \dfrac{p_0\gamma}{\rho_0}$

Wave equation for air:

$$\frac{\partial^2 \Phi}{\partial t^2} - c^2\nabla^2\Phi = 0$$

Here, $\quad c^2 = \left(\dfrac{\partial p}{\partial \rho}\right)_s$

$\mathbf{v} = -\nabla\Phi$

Note that, we also have:

$$\frac{\partial^2 \delta\rho}{\partial t^2} - c^2\nabla^2\delta\rho = 0$$

$$\frac{\partial^2 \delta p}{\partial t^2} - c^2\nabla^2\delta p = 0$$

Boundary values:

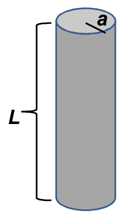
Impenetrable surface with normal $\hat{\mathbf{n}}$ moving at velocity $\mathbf{V}$:

$$\hat{\mathbf{n}}\cdot\mathbf{V} = \hat{\mathbf{n}}\cdot\delta\mathbf{v} = -\,\hat{\mathbf{n}}\cdot\nabla\Phi$$

Free surface:

$$\delta p = 0 \qquad \Rightarrow \rho_0\frac{\partial \Phi}{\partial t} = 0$$

Time harmonic standing waves in a pipe



$$\frac{\partial^2 \Phi}{\partial t^2} - c^2 \nabla^2 \Phi = 0$$

Boundary values :

At fixed surface : $\hat{\mathbf{n}} \cdot \nabla \Phi = 0$

At free surface : $\frac{\partial \Phi}{\partial t} = 0$

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$$\nabla^2 \Phi - \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} = 0 \qquad \text{Define : } k \equiv \frac{\omega}{c}$$

In cylindrical coordinates :

$$\Phi(r, \varphi, z, t) = R(r) F(\varphi) Z(z) e^{-i\omega t} \equiv R(r) F(\varphi) Z(z) e^{-ikct}$$

$$\nabla^2 = \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}$$

$$\left(\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2} + k^2 \right) \Phi(r, \varphi, z, t) = 0$$

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$$\left(\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2} + k^2 \right) \Phi(r, \varphi, z, t) = 0$$

$$\Phi(r, \varphi, z, t) = R(r) F(\varphi) Z(z) e^{-i\omega t}$$

$$F(\varphi) = e^{im\varphi}; \; F(\varphi) = F(\varphi + 2\pi N) \Rightarrow m = \text{integer}$$

$$Z(z) = e^{i\alpha z}; \; \alpha = \text{real plus other restrictions}$$

$$\left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{m^2}{r^2} - \alpha^2 + k^2 \right) R(r) = 0$$

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$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{m^2}{r^2}-\alpha^2+k^2\right)R(r)=0$$

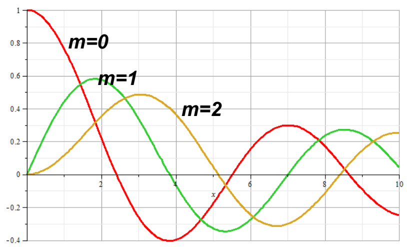
For $\quad k^2 \geq \alpha^2 \quad$ define $\quad \kappa^2 \equiv k^2-\alpha^2$

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{m^2}{r^2}+\kappa^2\right)R(r)=0$$

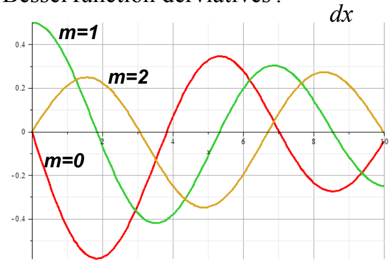
Cylinder surface boundary conditions : $\left.\frac{dR}{dr}\right|_{r=a}=0$

$\Rightarrow R(r)=J_m(\kappa r) \quad$ where for $\frac{dJ_m(x'_{mn})}{dx}=0,\ \kappa_{mn}=\frac{x'_{mn}}{a}$

Bessel functions : $J_m(x)$



Bessel function derviatives : $\frac{dJ_m(x)}{dx}$



Zeros of derivatives: m=0: 0.00000, 3.83171, 7.01559
m=1: 1.84118, 5.33144, 8.53632
m=2: 3.05424, 6.70613, 9.96947

Boundary condition for z=0, z=L:

For open - open pipe :

$$Z(0) = Z(L) = 0 \quad \Rightarrow Z(z) = \sin\left(\frac{p\pi z}{L}\right)$$

$$\Rightarrow \alpha_p = \frac{p\pi}{L}, \quad p = 1,2,3...$$

Resonant frequencies :

$$\frac{\omega^2}{c^2} = k^2 = \kappa_{mn}^2 + \alpha_p^2$$

$$k_{mnp}^2 = \left(\frac{x'_{mn}}{a}\right)^2 + \left(\frac{\pi p}{L}\right)^2$$

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Example

$$k_{mnp}^2 = \left(\frac{x'_{mn}}{a}\right)^2 + \left(\frac{\pi p}{L}\right)^2 = \left(\frac{\pi p}{L}\right)^2\left(1 + \left(\frac{L}{a}\right)^2\left(\frac{x'_{mn}}{\pi p}\right)^2\right)$$

$$\pi p = 3.14, 6.28, 9.42....$$

$$x'_{mn} = 0.00, 1.84, 3.05$$

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Alternate boundary condition for z=0, z=L:

For open - closed pipe :

$$\frac{dZ(0)}{dz} = Z(L) = 0 \quad \Rightarrow Z(z) = \cos\left(\frac{(2p+1)\pi z}{2L}\right)$$

$$\Rightarrow \alpha_p = \frac{(2p+1)\pi}{2L}, \quad p = 0,1,2,3...$$

$$k_{mnp}^2 = \left(\frac{x'_{mn}}{a}\right)^2 + \left(\frac{\pi(2p+1)}{2L}\right)^2$$

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Other solutions to wave equation:

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = 0$$

Plane wave solution :

$$\Phi(\mathbf{r},t) = Ae^{i\mathbf{k}\cdot\mathbf{r}-i\omega t} \quad \text{where} \quad k^2 = \left(\frac{\omega}{c}\right)^2$$