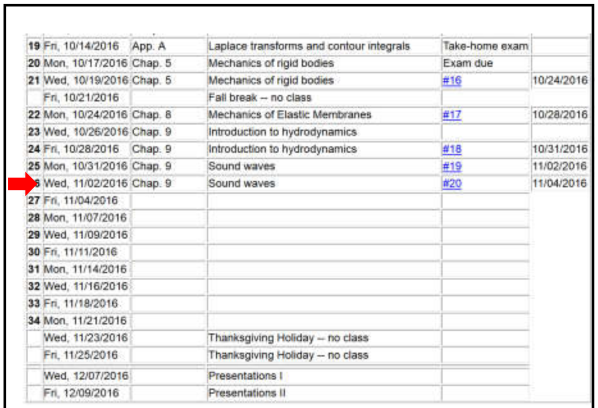
PHY 711 Classical Mechanics and Mathematical Methods
11-11:50 AM MWF Olin 107

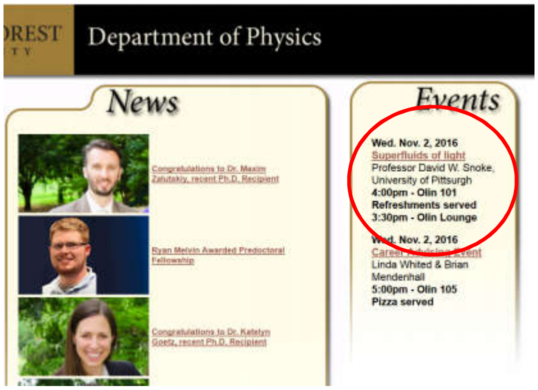
Plan for Lecture 26: Chap. 9 of F&W

Wave equation for sound in the linear approximation

1. Sound generation
2. Sound scattering



19	Fri, 10/14/2016	App. A	Laplace transforms and contour integrals	Take-home exam	
20	Mon, 10/17/2016	Chap. 5	Mechanics of rigid bodies	Exam due	
21	Wed, 10/19/2016	Chap. 5	Mechanics of rigid bodies	#16	10/24/2016
	Fri, 10/21/2016		Fall break -- no class		
22	Mon, 10/24/2016	Chap. 8	Mechanics of Elastic Membranes	#17	10/28/2016
23	Wed, 10/26/2016	Chap. 9	Introduction to hydrodynamics		
24	Fri, 10/28/2016	Chap. 9	Introduction to hydrodynamics	#18	10/31/2016
25	Mon, 10/31/2016	Chap. 9	Sound waves	#19	11/02/2016
6	Wed, 11/02/2016	Chap. 9	Sound waves	#20	11/04/2016
27	Fri, 11/04/2016				
28	Mon, 11/07/2016				
29	Wed, 11/09/2016				
30	Fri, 11/11/2016				
31	Mon, 11/14/2016				
32	Wed, 11/16/2016				
33	Fri, 11/18/2016				
34	Mon, 11/21/2016				
	Wed, 11/23/2016		Thanksgiving Holiday -- no class		
	Fri, 11/25/2016		Thanksgiving Holiday -- no class		
	Wed, 12/07/2016		Presentations I		
	Fri, 12/09/2016		Presentations II		



Department of Physics

News

Congratulations to Dr. Maxim Zalutskiy, recent Ph.D. Recipient

Ryan Melvin Awarded Predoctoral Fellowship

Congratulations to Dr. Katelyn Goetz, recent Ph.D. Recipient

Events

Wed. Nov. 2, 2016
Superfluids of light
Professor David W. Snoke, University of Pittsurgh
4:00pm - Olin 101
Refreshments served
3:30pm - Olin Lounge

Wed. Nov. 2, 2016
Career Advising Event
Linda Whited & Brian Mendenhall
5:00pm - Olin 105
Pizza served

Solutions to wave equation:

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = 0$$

Plane wave solution :

$$\Phi(\mathbf{r},t) = Ae^{i\mathbf{k}\cdot\mathbf{r}-i\omega t} \quad \text{where} \quad k^2 = \left(\frac{\omega}{c}\right)^2$$

Wave equation with source:

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -f(\mathbf{r},t)$$

Solution in terms of Green's function :

$$\Phi(\mathbf{r},t) = \int d^3r' \int dt'\, G(\mathbf{r}-\mathbf{r}', t-t') f(\mathbf{r}',t')$$

where

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G(\mathbf{r}-\mathbf{r}', t-t') = -\delta(\mathbf{r}-\mathbf{r}')\delta(t-t')$$

Wave equation with source -- continued:

We can show that :

$$G(\mathbf{r}-\mathbf{r}', t-t') = \frac{\delta\left(t' - \left(t \mp \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)\right)}{4\pi|\mathbf{r}-\mathbf{r}'|}$$

Derivation of Green's function for wave equation

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G(\mathbf{r}-\mathbf{r}', t-t') = -\delta(\mathbf{r}-\mathbf{r}')\delta(t-t')$$

Recall that

$$\delta(t-t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega(t-t')}d\omega$$

Derivation of Green's function for wave equation -- continued

$$\text{Define: } \quad \widetilde{G}(\mathbf{r},\omega) = \int_{-\infty}^{\infty} G(\mathbf{r},t)e^{i\omega t}dt$$

$$G(\mathbf{r},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\widetilde{G}(\mathbf{r},\omega)e^{-i\omega t}d\omega$$

$\widetilde{G}(\mathbf{r},\omega)$ must satisfy:

$$\left(\nabla^2 + k^2\right)\widetilde{G}(\mathbf{r}-\mathbf{r}',\omega) = -\delta(\mathbf{r}-\mathbf{r}') \quad \text{where } k^2 = \frac{\omega^2}{c^2}$$

Derivation of Green's function for wave equation -- continued

$$\left(\nabla^2 + k^2\right)\widetilde{G}(\mathbf{r}-\mathbf{r}',\omega) = -\delta(\mathbf{r}-\mathbf{r}')$$

Solution assuming isotropy in $\mathbf{r}-\mathbf{r}'$:

$$\widetilde{G}(\mathbf{r}-\mathbf{r}',\omega) = \frac{e^{\pm ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}$$

Check -- Define $R \equiv |\mathbf{r}-\mathbf{r}'|$ and for $R > 0$:

$$\left(\nabla^2 + k^2\right)\widetilde{G}(R,\omega) = \frac{1}{R}\frac{d^2}{dR^2}\left(R\widetilde{G}(R,\omega)\right) + k^2\widetilde{G}(R,\omega) = 0$$

Derivation of Green's function for wave equation -- continued

For $R>0$:

$$\left(\nabla^2+k^2\right)\widetilde{G}(R,\omega)=\frac{1}{R}\frac{d^2}{dR^2}\left(R\widetilde{G}(R,\omega)\right)+k^2\widetilde{G}(R,\omega)=0$$

$$\frac{d^2}{dR^2}\left(R\widetilde{G}(R,\omega)\right)+k^2\left(R\widetilde{G}(R,\omega)\right)=0$$

$$\left(R\widetilde{G}(R,\omega)\right)=A\,e^{ikR}+Be^{-ikR}$$

$$\Rightarrow\ \widetilde{G}(R,\omega)=\ A\frac{e^{ikR}}{R}+B\frac{e^{-ikR}}{R}$$

Derivation of Green's function for wave equation – continued
need to find *A* and *B*.

Note that: $\nabla^2\frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}=-\delta(\mathbf{r}-\mathbf{r}')$

$$\Rightarrow A=B=\frac{1}{4\pi}$$

$$\widetilde{G}(R,\omega)=\ \frac{e^{\pm ikR}}{4\pi R}$$

Derivation of Green's function for wave equation – continued

$$G(\mathbf{r}-\mathbf{r}',t-t')=\frac{1}{2\pi}\int_{-\infty}^{\infty}\widetilde{G}(\mathbf{r}-\mathbf{r}',\omega)e^{-i\omega(t-t')}d\omega$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{\pm ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}e^{-i\omega(t-t')}d\omega$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{\pm i\frac{\omega}{c}|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}e^{-i\omega(t-t')}d\omega$$

Derivation of Green's function for wave equation – continued

$$G(\mathbf{r}-\mathbf{r}',t-t')=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{\pm i\frac{\omega}{c}|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}e^{-i\omega(t-t')}d\omega$$

Noting that $\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega u}d\omega \;=\; \delta(u)$

$$\Rightarrow G(\mathbf{r}-\mathbf{r}',t-t')=\frac{\delta\left(t-\left(t'\mp\frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)\right)}{4\pi|\mathbf{r}-\mathbf{r}'|}$$

➔In order to solve an inhomogenous wave equation with a time harmonic forcing term, we can use the corresponding Green's function:

$$\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)=\frac{e^{\pm ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}$$

In fact, this Green's function is appropriate for solving equations with boundary conditions at infinity. For solving problems with surface boundary conditions where we know the boundary values or their gradients, the Green's function must be modified.

Green's theorem

Consider two functions $h(\mathbf{r})$ and $g(\mathbf{r})$

Note that : $\int_V \left(h\nabla^2 g - g\nabla^2 h\right)d^3r = \oint_S \left(h\nabla g - g\nabla h\right)\cdot\hat{\mathbf{n}}\,d^2r$

$$\nabla^2\tilde{\Phi}+k^2\tilde{\Phi}=-\tilde{f}(\mathbf{r},\omega)$$

$$\left(\nabla^2+k^2\right)\tilde{G}(\mathbf{r}-\mathbf{r}',\omega)=-\delta(\mathbf{r}-\mathbf{r}')$$

$$h\leftrightarrow\tilde{\Phi};\qquad g\leftrightarrow\tilde{G}$$

$$\int_V\left(\tilde{\Phi}(\mathbf{r},\omega)\delta(\mathbf{r}-\mathbf{r}')-\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)f(\mathbf{r},\omega)\right)d^3r=$$

$$\oint_S\left(\tilde{\Phi}(\mathbf{r},\omega)\nabla\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)-\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)\nabla\tilde{\Phi}(\mathbf{r},\omega)\right)\cdot\hat{\mathbf{n}}\,d^2r$$

$$\int_V \left(\tilde{\Phi}(\mathbf{r},\omega)\delta(\mathbf{r}-\mathbf{r}') - \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) f(\mathbf{r},\omega)\right) d^3r =$$

$$\oint_S \left(\tilde{\Phi}(\mathbf{r},\omega)\nabla\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) - \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)\nabla\tilde{\Phi}(\mathbf{r},\omega)\right)\cdot\hat{\mathbf{n}}d^2r$$

Exchanging $\mathbf{r}\leftrightarrow\mathbf{r}'$:

$$\int_V \left(\tilde{\Phi}(\mathbf{r}',\omega)\delta(\mathbf{r}-\mathbf{r}') - \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) f(\mathbf{r}',\omega)\right) d^3r' =$$

$$\oint_S \left(\tilde{\Phi}(\mathbf{r}',\omega)\nabla\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) - \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)\nabla\tilde{\Phi}(\mathbf{r}',\omega)\right)\cdot\hat{\mathbf{n}}d^2r'$$

If the integration volume V includes the point $\mathbf{r}=\mathbf{r}'$:

$$\tilde{\Phi}(\mathbf{r},\omega) = \int_V \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) f(\mathbf{r}',\omega) d^3r' +$$

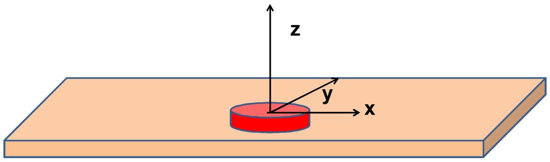
$$\oint_S \left(\tilde{\Phi}(\mathbf{r}',\omega)\nabla\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) - \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)\nabla\tilde{\Phi}(\mathbf{r}',\omega)\right)\cdot\hat{\mathbf{n}}d^2r'$$

Wave equation with source:

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -f(\mathbf{r},t)$$

Example:

$f(\mathbf{r},t) \Rightarrow$ time harmonic piston of radius a, amplitude $\varepsilon\hat{\mathbf{z}}$ can be represented as boundary value of $\Phi(\mathbf{r},t)$



Treatment of boundary values for time-harmonic force:

$$\tilde{\Phi}(\mathbf{r},\omega) = \int_V \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)\tilde{f}(\mathbf{r}',\omega) d^3r' +$$

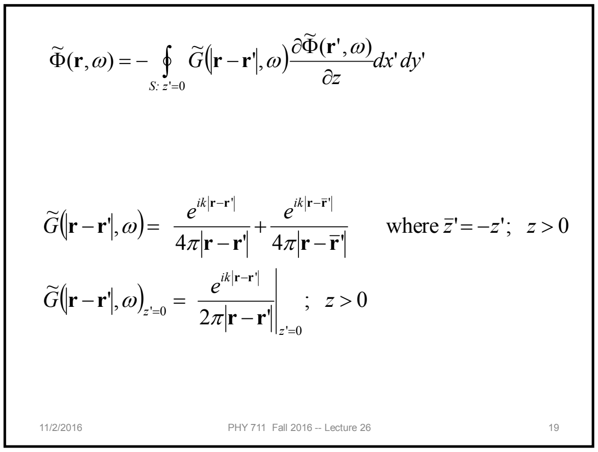
$$\oint_S \left(\tilde{\Phi}(\mathbf{r}',\omega)\nabla'\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) - \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)\nabla'\tilde{\Phi}(\mathbf{r}',\omega)\right)\cdot\hat{\mathbf{n}}d^2r'$$

Boundary values for our example:

$$\left(\frac{\partial\tilde{\Phi}}{\partial z}\right)_{z=0} = \begin{cases} 0 & \text{for } x^2+y^2 > a^2 \\ i\omega\varepsilon a & \text{for } x^2+y^2 < a^2 \end{cases}$$

Note: Need Green's function with vanishing gradient at $z=0$:

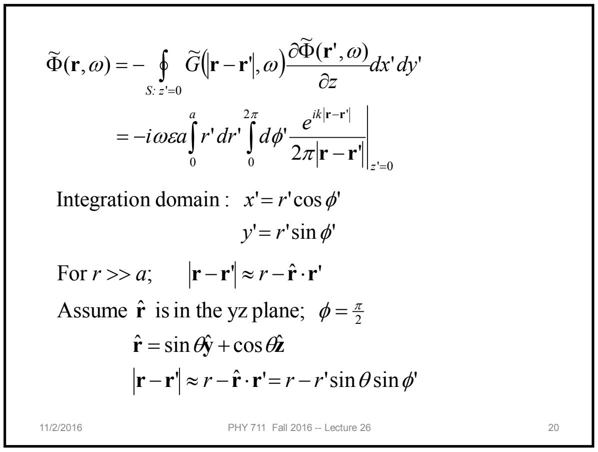
$$\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) = \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} + \frac{e^{ik|\mathbf{r}-\bar{\mathbf{r}}'|}}{4\pi|\mathbf{r}-\bar{\mathbf{r}}'|} \qquad \text{where } \bar{z}' = -z'; \quad z > 0$$



$$\tilde{\Phi}(\mathbf{r},\omega) = -\oint_{S:\,z'=0} \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)\frac{\partial\tilde{\Phi}(\mathbf{r}',\omega)}{\partial z}dx'\,dy'$$

$$\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) = \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} + \frac{e^{ik|\mathbf{r}-\bar{\mathbf{r}}'|}}{4\pi|\mathbf{r}-\bar{\mathbf{r}}'|} \qquad \text{where } \bar{z}' = -z'; \quad z > 0$$

$$\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)_{z'=0} = \left.\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{2\pi|\mathbf{r}-\mathbf{r}'|}\right|_{z'=0}; \quad z > 0$$



$$\tilde{\Phi}(\mathbf{r},\omega) = -\oint_{S:\,z'=0} \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)\frac{\partial\tilde{\Phi}(\mathbf{r}',\omega)}{\partial z}dx'\,dy'$$

$$= -i\omega\varepsilon a\int_0^a r'\,dr'\int_0^{2\pi} d\phi' \left.\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{2\pi|\mathbf{r}-\mathbf{r}'|}\right|_{z'=0}$$

Integration domain : $x' = r'\cos\phi'$

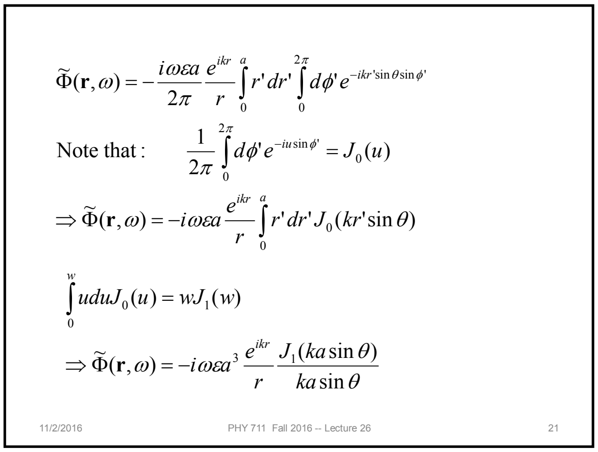
$y' = r'\sin\phi'$

For $r >> a$; $|\mathbf{r}-\mathbf{r}'| \approx r - \hat{\mathbf{r}}\cdot\mathbf{r}'$

Assume $\hat{\mathbf{r}}$ is in the yz plane; $\phi = \frac{\pi}{2}$

$$\hat{\mathbf{r}} = \sin\theta\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}$$

$$|\mathbf{r}-\mathbf{r}'| \approx r - \hat{\mathbf{r}}\cdot\mathbf{r}' = r - r'\sin\theta\sin\phi'$$



$$\tilde{\Phi}(\mathbf{r},\omega) = -\frac{i\omega\varepsilon a}{2\pi}\frac{e^{ikr}}{r}\int_0^a r'\,dr'\int_0^{2\pi} d\phi'\, e^{-ikr'\sin\theta\sin\phi'}$$

Note that : $\frac{1}{2\pi}\int_0^{2\pi} d\phi'\, e^{-iu\sin\phi'} = J_0(u)$

$$\Rightarrow \tilde{\Phi}(\mathbf{r},\omega) = -i\omega\varepsilon a\frac{e^{ikr}}{r}\int_0^a r'\,dr'\,J_0(kr'\sin\theta)$$

$$\int_0^w u\,du\,J_0(u) = wJ_1(w)$$

$$\Rightarrow \tilde{\Phi}(\mathbf{r},\omega) = -i\omega\varepsilon a^3\frac{e^{ikr}}{r}\frac{J_1(ka\sin\theta)}{ka\sin\theta}$$

Energy flux: $\mathbf{j}_e = \delta\mathbf{v}p$

Taking time average: $\langle \mathbf{j}_e \rangle = \frac{1}{2}\Re\left(\delta\mathbf{v}p^*\right)$

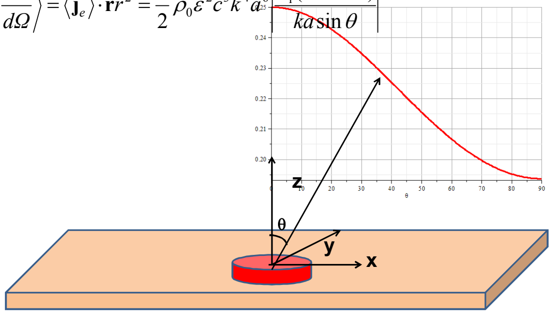
$$= \tfrac{1}{2}\rho_0\Re\left((-\nabla\Phi)(-i\omega\Phi)^*\right)$$

Time averaged power per solid angle:

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \langle \mathbf{j}_e \rangle \cdot \hat{\mathbf{r}} r^2 = \frac{1}{2}\rho_0 \varepsilon^2 c^3 k^4 a^6 \left| \frac{J_1(ka\sin\theta)}{ka\sin\theta} \right|^2$$

Time averaged power per solid angle:

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \langle \mathbf{j}_e \rangle \cdot \hat{\mathbf{r}} r^2 = \frac{1}{2}\rho_0 \varepsilon^2 c^3 k^4 a^6 \left| \frac{J_1(ka\sin\theta)}{ka\sin\theta} \right|^2$$



Scattering of sound waves –
for example, from a rigid cylinder

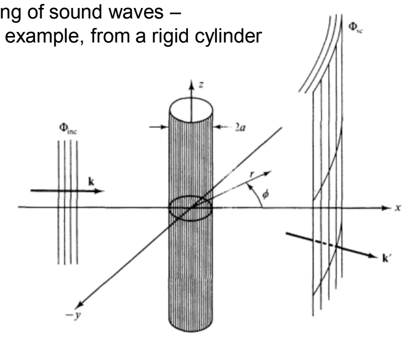


Figure 51.8 Scattering from a rigid cylinder.

Figure from Fetter and Walecka pg. 337

Scattering of sound waves –
for example, from a rigid cylinder

Velocity potential --

$$\Phi(\mathbf{r}) = \Phi_{inc}(\mathbf{r}) + \Phi_{sc}(\mathbf{r}) \qquad \Phi_{inc}(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}}$$

Helmholz equation in cylindrical coordinates:

$$\left(\nabla^2 + k^2\right)\Phi(\mathbf{r}) = 0 = \left(\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial \phi^2} + \frac{\partial}{\partial z^2} + k^2\right)\Phi(\mathbf{r})$$

Assume: $\Phi(\mathbf{r}) = \sum_{m=-\infty}^{\infty} e^{im\phi} R_m(r)$

where $\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m^2}{r^2} + k^2\right) R_m(r) = 0$

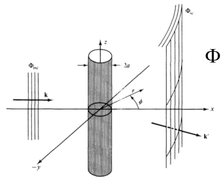


Figure 51.8 Scattering from a rigid cylinder.

$$\Phi_{inc}(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}} = e^{ikr\cos\phi} = \sum_{m=-\infty}^{\infty} i^m e^{im\phi} J_m(kr)$$

$\Phi_{sc}(\mathbf{r}) = \sum_{m=-\infty}^{\infty} C_m e^{im\phi} H_m(kr)$ where Hankel function

represents an outgoing wave : $H_m(kr) = J_m(kr) + iN_m(kr)$

Boundary condition at $r = a$: $\left.\frac{\partial \Phi}{\partial r}\right|_{r=a} = 0$

$$\Rightarrow i^m J'_m(ka) + C_m H'_m(ka) = 0 \qquad C_m = -i^m \frac{J'_m(ka)}{H'_m(ka)}$$

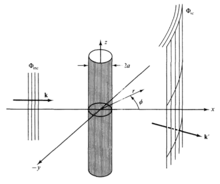


Figure 51.8 Scattering from a rigid cylinder.

$$\Phi_{sc}(\mathbf{r}) = -\sum_{m=-\infty}^{\infty} i^m \frac{J'_m(ka)}{H'_m(ka)} e^{im\phi} H_m(kr)$$

Asymptotic form:

$$i^m H_m(kr) \underset{kr\to\infty}{\approx} \sqrt{\frac{2}{\pi kr}} e^{i(kr-\pi/4)}$$

$$\Phi_{sc}(\mathbf{r}) \underset{kr\to\infty}{\approx} f(\phi)\sqrt{\frac{1}{r}} e^{ikr} = -\sum_{m=-\infty}^{\infty} \frac{J'_m(ka)}{H'_m(ka)} e^{im\phi} \sqrt{\frac{2}{\pi kr}} e^{i(kr-\pi/4)}$$

$$\Rightarrow f(\phi) = -\sqrt{\frac{2}{\pi k}} \sum_{m=-\infty}^{\infty} \frac{J'_m(ka)}{H'_m(ka)} e^{i(m\phi-\pi/4)}$$

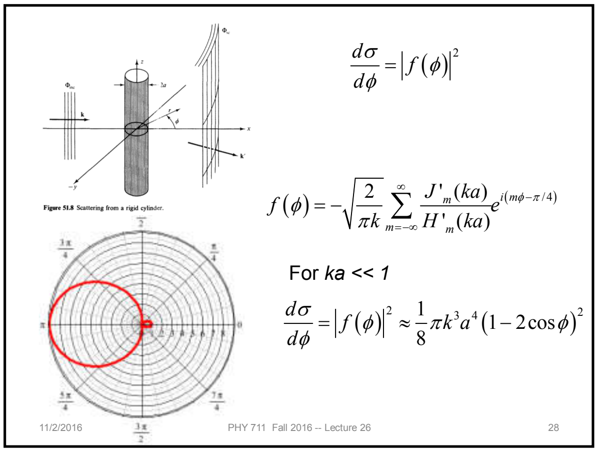


Figure 51.8 Scattering from a rigid cylinder.

$$\frac{d\sigma}{d\phi} = \left|f(\phi)\right|^2$$

$$f(\phi) = -\sqrt{\frac{2}{\pi k}} \sum_{m=-\infty}^{\infty} \frac{J'_m(ka)}{H'_m(ka)} e^{i(m\phi-\pi/4)}$$

For *ka << 1*

$$\frac{d\sigma}{d\phi} = \left|f(\phi)\right|^2 \approx \frac{1}{8}\pi k^3 a^4 \left(1-2\cos\phi\right)^2$$