PHY 711 Classical Mechanics and Mathematical Methods
11-11:50 AM MWF Olin 107

Plan for Lecture 27: Chap. 9 of F&W

Wave equation for sound –

non-linear effects

1. **Non-linear effects in traveling sound waves**
2. **Shocking effects**

19	Fri, 10/14/2016	App. A	Laplace transforms and contour integrals	Take-home exam	
20	Mon, 10/17/2016	Chap. 5	Mechanics of rigid bodies	Exam due	
21	Wed, 10/19/2016	Chap. 5	Mechanics of rigid bodies	#16	10/24/2016
	Fri, 10/21/2016		Fall break -- no class		
22	Mon, 10/24/2016	Chap. 8	Mechanics of Elastic Membranes	#17	10/28/2016
23	Wed, 10/26/2016	Chap. 9	Introduction to hydrodynamics		
24	Fri, 10/28/2016	Chap. 9	Introduction to hydrodynamics	#18	10/31/2016
25	Mon, 10/31/2016	Chap. 9	Sound waves	#19	11/02/2016
26	Wed, 11/02/2016	Chap. 9	Sound waves	#20	11/04/2016
27	Fri, 11/04/2016	Chap. 9	Non-linear sound	#21	11/07/2016
28	Mon, 11/07/2016				
29	Wed, 11/09/2016				
30	Fri, 11/11/2016				
31	Mon, 11/14/2016				
32	Wed, 11/16/2016				
33	Fri, 11/18/2016				
34	Mon, 11/21/2016				
	Wed, 11/23/2016		Thanksgiving Holiday -- no class		
	Fri, 11/25/2016		Thanksgiving Holiday -- no class		
	Wed, 12/07/2016		Presentations I		
	Fri, 12/09/2016		Presentations II		

Effects of nonlinearities in fluid equations
-- one dimensional case

Newton - Euler equation of motion :

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{f}_{applied} - \frac{\nabla p}{\rho}$$

Continuity equation : $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$

Assume spatial variation confined to x direction ;
assume that $\mathbf{v} = v\hat{\mathbf{x}}$ and $\mathbf{f}_{applied} = 0$.

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x} + \rho\frac{\partial v}{\partial x} = 0$$

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}+\frac{1}{\rho}\frac{\partial p}{\partial x}=0$$

$$\frac{\partial \rho}{\partial t}+v\frac{\partial \rho}{\partial x}+\rho\frac{\partial v}{\partial x}=0$$

Expressing p in terms of ρ: $p=p(\rho)$

$$\frac{\partial p}{\partial x}=\frac{\partial p}{\partial \rho}\frac{\partial \rho}{\partial x}\equiv c^2(\rho)\frac{\partial \rho}{\partial x} \qquad \text{where } \frac{\partial p}{\partial \rho}\equiv c^2(\rho)$$

For adiabatic ideal gas: $\frac{dp}{p}=\gamma\frac{d\rho}{\rho}$ $\qquad p=p_0\left(\frac{\rho}{\rho_0}\right)^{\gamma}$

$$c^2(\rho)=\frac{\gamma p}{\rho}=c_0^2\left(\frac{\rho}{\rho_0}\right)^{\gamma-1} \qquad \text{where } c_0^2\equiv\frac{\gamma p_0}{\rho_0}$$

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}+\frac{c^2(\rho)}{\rho}\frac{\partial \rho}{\partial x}=0$$

$$\frac{\partial \rho}{\partial t}+v\frac{\partial \rho}{\partial x}+\rho\frac{\partial v}{\partial x}=0$$

Expressing variation of v in terms of $v(\rho)$:

$$\frac{\partial v}{\partial \rho}\frac{\partial \rho}{\partial t}+v\frac{\partial v}{\partial \rho}\frac{\partial \rho}{\partial x}+\frac{c^2(\rho)}{\rho}\frac{\partial \rho}{\partial x}=0$$

$$\frac{\partial \rho}{\partial t}+v\frac{\partial \rho}{\partial x}+\rho\frac{\partial v}{\partial \rho}\frac{\partial \rho}{\partial x}=0$$

Some more algebra :

From Euler equation : $\frac{\partial v}{\partial \rho}\left(\frac{\partial \rho}{\partial t}+v\frac{\partial \rho}{\partial x}\right)+\frac{c^2(\rho)}{\rho}\frac{\partial \rho}{\partial x}=0$

From continuity equation : $\frac{\partial \rho}{\partial t}+v\frac{\partial \rho}{\partial x}=-\rho\frac{\partial v}{\partial \rho}\frac{\partial \rho}{\partial x}$

Combined equation : $\frac{\partial v}{\partial \rho}\left(-\rho\frac{\partial v}{\partial \rho}\frac{\partial \rho}{\partial x}\right)+\frac{c^2(\rho)}{\rho}\frac{\partial \rho}{\partial x}=0$

$$\Rightarrow\left(\frac{\partial v}{\partial \rho}\right)^2=\frac{c^2(\rho)}{\rho^2} \qquad \frac{\partial v}{\partial \rho}=\pm\frac{c}{\rho}$$

$$\Rightarrow\frac{\partial \rho}{\partial t}+(v\pm c)\frac{\partial \rho}{\partial x}=0$$

Assuming adiabatic process : $c^2 = c_0^2 \left(\frac{\rho}{\rho_0}\right)^{\gamma-1}$ $\quad c_0^2 = \frac{\gamma p_0}{\rho_0}$

$$\frac{\partial v}{\partial \rho} = \frac{dv}{d\rho} = \pm \frac{c}{\rho} \qquad \Rightarrow v = \pm c_0 \int_{\rho_0}^{\rho} \left(\frac{\rho'}{\rho_0}\right)^{(\gamma-1)/2} \frac{d\rho'}{\rho'}$$

$$\Rightarrow v = \pm \frac{2c_0}{\gamma - 1}\left(\left(\frac{\rho}{\rho_0}\right)^{(\gamma-1)/2} - 1\right)$$

$$\Rightarrow c = c_0 \left(\frac{\rho}{\rho_0}\right)^{(\gamma-1)/2}$$

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Summary :

$$\frac{dv}{d\rho} = \pm \frac{c}{\rho}$$

$$\frac{\partial \rho}{\partial t} + (v \pm c)\frac{\partial \rho}{\partial x} = 0$$

Assuming adiabatic process : $c^2 = c_0^2 \left(\frac{\rho}{\rho_0}\right)^{\gamma-1}$ $\quad c_0^2 = \frac{\gamma p_0}{\rho_0}$

$$c = c_0 \left(\frac{\rho}{\rho_0}\right)^{(\gamma-1)/2} \qquad v = \pm \frac{2c_0}{\gamma - 1}\left(\left(\frac{\rho}{\rho_0}\right)^{(\gamma-1)/2} - 1\right)$$

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Traveling wave solution:

Assume : $\rho = \rho_0 + f(x - u(\rho)t)$

Need to find self - consistent equations for propagation velocity $u(\rho)$ using equations

From previous derivations : $\frac{\partial \rho}{\partial t} + (v \pm c)\frac{\partial \rho}{\partial x} = 0$

Apparently : $u(\rho) \Leftrightarrow v \pm c$

For adiabatic ideal gas and + signs :

$$u = v + c = c_0 \left(\frac{\gamma+1}{\gamma-1}\left(\frac{\rho}{\rho_0}\right)^{(\gamma-1)/2} - \frac{2}{\gamma - 1}\right)$$

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Traveling wave solution -- continued:

$$\frac{\partial \rho}{\partial t} + (v \pm c)\frac{\partial \rho}{\partial x} = 0$$

Assume : $\rho = \rho_0 + f(x - u(\rho)t) = \rho_0 + f(x - (v \pm c)t)$

For adiabatic ideal gas and + signs :

$$u = v + c = c_0\left(\frac{\gamma+1}{\gamma-1}\left(\frac{\rho}{\rho_0}\right)^{(\gamma-1)/2} - \frac{2}{\gamma-1}\right)$$

Solution in linear approxiation :

$$u = v + c \approx v_0 + c_0 = c_0\left(\frac{\gamma+1}{\gamma-1} - \frac{2}{\gamma-1}\right) = c_0$$

$$\Rightarrow \rho = \rho_0 + f(x - c_0 t)$$

Traveling wave solution -- full non-linear case:

Visualization for particular waveform: $\rho = \rho_0 + f(\underbrace{x - u(\rho)t}_{w})$

Assume: $f(w) \equiv \rho_0 s(w)$

$$\frac{\rho}{\rho_0} = 1 + s(x - ut)$$

For adiabatic ideal gas:

$$u = c_0\left(\frac{\gamma+1}{\gamma-1}\left(\frac{\rho}{\rho_0}\right)^{(\gamma-1)/2} - \frac{2}{\gamma-1}\right)$$

$$u = c_0\left(\frac{\gamma+1}{\gamma-1}\left(1 + s(x - ut)\right)^{(\gamma-1)/2} - \frac{2}{\gamma-1}\right)$$

Visualization continued:

$$u = c_0\left(\frac{\gamma+1}{\gamma-1}\left(1 + s(x - ut)\right)^{(\gamma-1)/2} - \frac{2}{\gamma-1}\right)$$

Plot $s(x - ut)$ for fixed t, as a function of x :

Let $w = x - ut$

$$x = w + ut = w + u(w)t \equiv x(w,t)$$

$$u(w) = c_0\left(\frac{\gamma+1}{\gamma-1}\left(1 + s(w)\right)^{(\gamma-1)/2} - \frac{2}{\gamma-1}\right)$$

Parametric equations: plot $s(w)$ vs $x(w,t)$ for range of w

Summary

$$\frac{\partial \rho}{\partial t} + u(\rho)\frac{\partial \rho}{\partial x} = 0$$

Solution: $\rho = \rho_0 + f(x - u(\rho)t) = \rho_0\left(1 + s(x - u(\rho)t)\right)$

For linear case: $u(\rho) = c_0$

For non-linear case: $u(\rho) = c_0\left(\frac{\gamma+1}{\gamma-1}\left(1+s(x-ut)\right)^{(\gamma-1)/2} - \frac{2}{\gamma-1}\right)$

Plot $s(x-ut)$ for fixed t, as a function of x:

Let $w = x - ut \quad \Rightarrow \quad x = w + ut = w + u(w)t \equiv x(w,t)$

$$u(w) = c_0\left(\frac{\gamma+1}{\gamma-1}\left(1+s(w)\right)^{(\gamma-1)/2} - \frac{2}{\gamma-1}\right)$$

Parametric equations: plot $s(w)$ vs $x(w,t)$ for range of w

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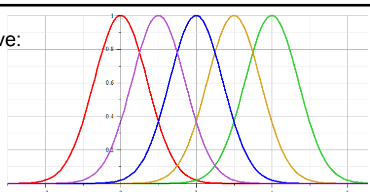
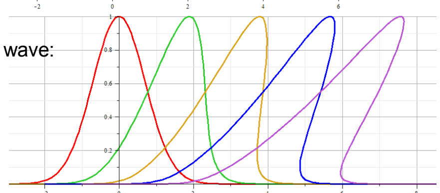
Linear wave:



Non-linear wave:



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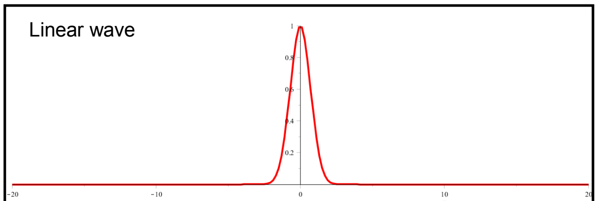
Linear wave



Non-linear wave

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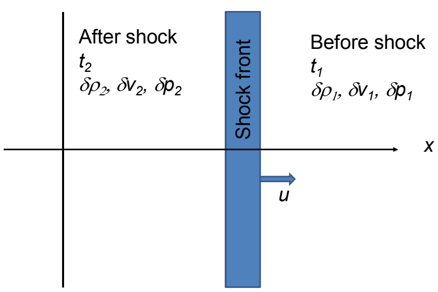
Analysis of shock wave
Plots of $\delta\rho$

Solution becomes unphysical

Analysis of shock wave -- continued



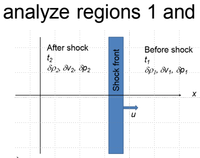
Analysis of shock wave – continued

While analysis in the shock region is complicated, we can use conservation laws to analyze regions 1 and 2

Assume $\rho(x,t)=\rho(x-ut)$

$$p(x,t)=p(x-ut)$$

$$v(x,t)=v(x-ut)$$



Continuity equation:

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho v)}{\partial x}=0=\frac{\partial(\rho v-\rho u)}{\partial x} \quad \Rightarrow (v_2-u)\rho_2=(v_1-u)\rho_1$$

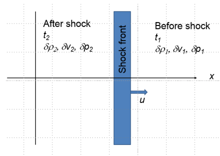
Conservation of energy and momentum:

$$\Rightarrow p_2+\rho_2(v_2-u)^2=p_1+\rho_1(v_1-u)^2$$

$$\Rightarrow \epsilon_2+\frac{1}{2}(v_2-u)^2+\frac{p_2}{\rho_2}=\epsilon_1+\frac{1}{2}(v_1-u)^2+\frac{p_1}{\rho_1}$$

Analysis of shock wave – continued

For adiabatic ideal gas, also considering energy and momentum conservation:



$$\frac{\rho_2}{\rho_1} = \frac{\frac{\gamma+1}{\gamma-1}\frac{p_2}{p_1}+1}{\frac{\gamma+1}{\gamma-1}+\frac{p_2}{p_1}} \leq \frac{\gamma+1}{\gamma-1}$$

Velocity relationships:

$$\frac{(v_1-u)^2}{c_1^2} = \frac{1}{2\gamma}\left(\gamma-1+(\gamma+1)\frac{p_2}{p_1}\right) \qquad \frac{(v_2-u)^2}{c_2^2} = \frac{1}{2\gamma}\left(\gamma-1+(\gamma+1)\frac{p_1}{p_2}\right)$$

where $c_1^2 \equiv \frac{\gamma p_1}{\rho_1}$ and $c_2^2 \equiv \frac{\gamma p_2}{\rho_2}$

For a strong shock:

$$\frac{(v_1-u)^2}{c_1^2} \rightarrow \frac{(\gamma+1)}{2\gamma}\frac{p_2}{p_1} \qquad \frac{(v_2-u)^2}{c_2^2} \rightarrow \frac{(\gamma-1)}{2\gamma}$$

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Analysis of shock wave – continued

For adiabatic ideal gas, entropy considerations::

Internal energy density: $\varepsilon = \frac{p}{(\gamma-1)\rho} = C_V T$

First law of thermo: $d\varepsilon = Tds - pd\left(\frac{1}{\rho}\right)$

$$ds = \frac{1}{T}\left(d\left(\frac{p}{(\gamma-1)\rho}\right) - pd\left(\frac{1}{\rho}\right)\right) = C_V d\ln\left(\frac{p}{\rho^\gamma}\right)$$

$$s = C_V \ln\left(\frac{p}{\rho^\gamma}\right) + (\text{constant})$$

$$s_2 - s_1 = C_V \ln\left(\frac{p_2}{p_1}\left(\frac{\rho_1}{\rho_2}\right)^\gamma\right) \qquad 0 < s_2 - s_1 < C_V\left(\ln\left(\frac{p_2}{p_1}\right) - \gamma\ln\left(\frac{\gamma+1}{\gamma-1}\right)\right)$$

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