PHY 711 Classical Mechanics and Mathematical Methods
11-11:50 AM MWF Olin 107

Plan for Lecture 29:

Chapter 10 in F & W: Surface waves

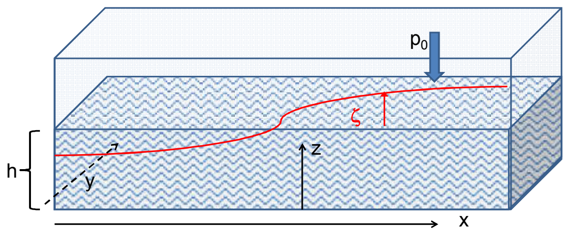
1. **Water waves in a channel**
2. **Wave-like solutions; wave speed**

21	Wed, 10/19/2016	Chap. 5	Mechanics of rigid bodies	#16	10/24/2016
	Fri, 10/21/2016		Fall break -- no class		
22	Mon, 10/24/2016	Chap. 8	Mechanics of Elastic Membranes	#17	10/28/2016
23	Wed, 10/26/2016	Chap. 9	Introduction to hydrodynamics		
24	Fri, 10/28/2016	Chap. 9	Introduction to hydrodynamics	#18	10/31/2016
25	Mon, 10/31/2016	Chap. 9	Sound waves	#19	11/02/2016
26	Wed, 11/02/2016	Chap. 9	Sound waves	#20	11/04/2016
27	Fri, 11/04/2016	Chap. 9	Non-linear sound	#21	11/07/2016
28	Mon, 11/07/2016	Chap. 10	Surface waves in fluids		
29	Wed, 11/09/2016				
30	Fri, 11/11/2016				
31	Mon, 11/14/2016				
32	Wed, 11/16/2016				
33	Fri, 11/18/2016				
34	Mon, 11/21/2016				
	Wed, 11/23/2016		Thanksgiving Holiday -- no class		
	Fri, 11/25/2016		Thanksgiving Holiday -- no class		
	Wed, 12/07/2016		Presentations I		
	Fri, 12/09/2016		Presentations II		

Physics of incompressible fluids and their surfaces

Reference: Chapter 10 of Fetter and Walecka

Consider a container of water with average height h and surface h+ζ(x,y,t); (h ←→ z_0 on some of the slides)



Euler's equation for incompressible fluid:

$$\frac{d\mathbf{v}}{dt} = f_{applied} - \frac{\nabla p}{\rho} = -g\hat{\mathbf{z}} - \frac{\nabla p}{\rho}$$

Assume that $v_z << v_x, v_y$ $\quad \Rightarrow -g - \frac{1}{\rho}\frac{\partial p}{\partial z} \approx 0$

$$\Rightarrow p(x,y,z,t) = p_0 + \rho g\left(\zeta(x,y,t) + h - z\right)$$

Horizontal fluid motions (keeping leading terms):

$$\frac{dv_x}{dt} \approx \frac{\partial v_x}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} = -g\frac{\partial \zeta}{\partial x}$$

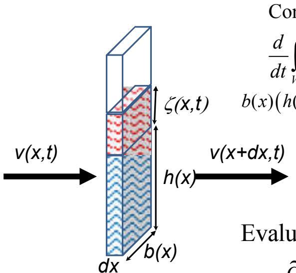
$$\frac{dv_y}{dt} \approx \frac{\partial v_y}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial y} = -g\frac{\partial \zeta}{\partial y}$$

Consider a surface *ζ(x,t)* wave moving in the *x*-direction in a channel of width *b(x)* and height *h(x)*:

Continuity condition in integral form:

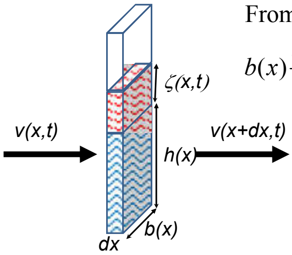
$$\frac{d}{dt}\int_V \rho dV + \int_A \rho \mathbf{v}\cdot d\mathbf{A} = 0$$

$$b(x)\left(h(x)+\zeta(x,t)\right)dx \qquad b(x)\left(h(x)+\zeta(x,t)\right)\hat{\mathbf{x}}$$



Evaluating continuity condition:

$$b(x)\frac{\partial \zeta}{\partial t} = -\frac{\partial}{\partial x}\left(h(x)b(x)v(x,t)\right)$$



From continuity condition:

$$b(x)\frac{\partial \zeta}{\partial t} = -\frac{\partial}{\partial x}\left(h(x)b(x)v(x,t)\right)$$

Example (Problem 10.3):

$$b(x) = b_0 \qquad h(x) = \kappa x$$

$$b_0 \frac{\partial \zeta}{\partial t} = -\frac{\partial}{\partial x}\left((\kappa x) b_0 v(x,t)\right)$$

$$\frac{\partial \zeta}{\partial t} = -\kappa\left(v + x\frac{\partial v}{\partial x}\right)$$

From Newton-Euler equation:

$$\frac{dv}{dt} \approx \frac{\partial v}{\partial t} = -g\frac{\partial \zeta}{\partial x}$$

Example continued

$$\frac{\partial \zeta}{\partial t} = -\kappa\left(v + x\frac{\partial v}{\partial x}\right) \quad \Rightarrow \quad \frac{\partial^2 \zeta}{\partial t^2} = -\kappa\left(\frac{\partial v}{\partial t} + x\frac{\partial^2 v}{\partial x \partial t}\right)$$

$$\frac{\partial v}{\partial t} = -g\frac{\partial \zeta}{\partial x} \quad \Rightarrow \quad \frac{\partial^2 \zeta}{\partial t^2} = \kappa g\left(\frac{\partial \zeta}{\partial x} + x\frac{\partial^2 \zeta}{\partial x^2}\right)$$

It can be shown that a solution can take the form:

$$\zeta(x,t) = CJ_0\left(\frac{2\omega}{\sqrt{\kappa g}}\sqrt{x}\right)\cos(\omega t)$$

Note that $J_0(u)$ satisfies the equation: $\left(\frac{d^2}{du^2} + \frac{1}{u}\frac{d}{du} + 1\right)J_0(u) = 0$

Therefore $J_0(w\sqrt{x})$ satisfies the equation:

$$\left(\frac{4x}{w^2}\frac{d^2}{dx^2} + \frac{4}{w^2}\frac{d}{dx} + 1\right)J_0\left(w\sqrt{x}\right) = 0$$

Example continued

$$\frac{\partial^2 \zeta}{\partial t^2} = \kappa g\left(\frac{\partial \zeta}{\partial x} + x\frac{\partial^2 \zeta}{\partial x^2}\right)$$

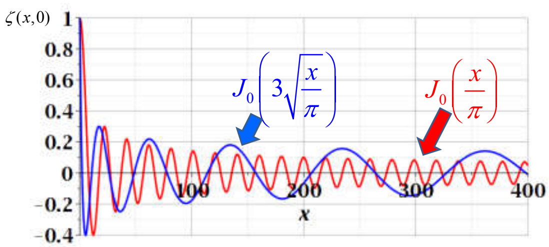
Since $J_0(w\sqrt{x})$ satisfies the equation:

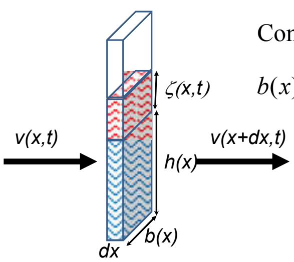
$$\left(\frac{4x}{w^2}\frac{d^2}{dx^2} + \frac{4}{w^2}\frac{d}{dx} + 1\right)J_0\left(w\sqrt{x}\right) = 0$$

$$\Rightarrow \zeta(x,t) = CJ_0\left(w\sqrt{x}\right)\cos(\omega t)$$

where $w = \frac{2\omega}{\sqrt{\kappa g}}$

$$\zeta(x,t) = CJ_0\left(\frac{2\omega}{\sqrt{\kappa g}}\sqrt{x}\right)\cos(\omega t)$$





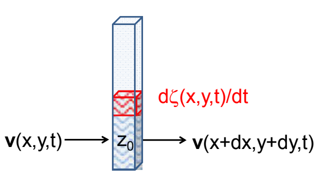
Continuity condition:

$$b(x)\frac{\partial \zeta}{\partial t} = -\frac{\partial}{\partial x}\big(h(x)b(x)v(x,t)\big)$$

Special case, where *b* and *h* are constant --

For constant *b* and *h*:

$$\frac{\partial \zeta}{\partial t} = -h\frac{\partial}{\partial x}\big(v(x,t)\big)$$



Continuity condition for flow of incompressible fluid:

$$\frac{\partial \zeta}{\partial t} + h\nabla\cdot\mathbf{v} = 0$$

From horizontal flow relations: $\frac{\partial \mathbf{v}}{\partial t} = -g\nabla\zeta$

Equation for surface function: $\frac{\partial^2 \zeta}{\partial t^2} - gh\nabla^2\zeta = 0$

For uniform channel:

Surface wave equation:

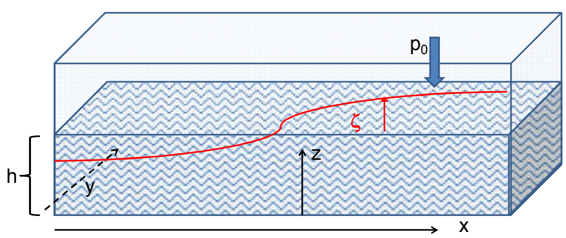
$$\frac{\partial^2 \zeta}{\partial t^2} - c^2 \nabla^2 \zeta = 0 \qquad\qquad c^2 = gh$$

More complete analysis finds:

$$c^2 = \frac{g}{k}\tanh(kh) \qquad \text{where } k = \frac{2\pi}{\lambda}$$

More details: -- recall setup --

Consider a container of water with average height h and surface h+ζ(x,y,t)



Equations describing fluid itself (without boundaries)

Euler's equation for incompressible fluid:

$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} = \frac{\partial \mathbf{v}}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) + \mathbf{v}\times\left(\nabla\times\mathbf{v}\right) = -\nabla U - \frac{\nabla p}{\rho}$$

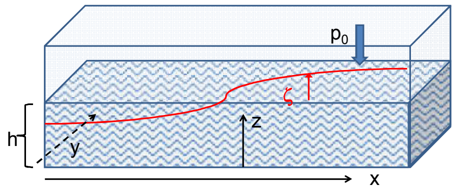
Assume that $\nabla\times\mathbf{v} = 0$ (irrotational flow) $\Rightarrow \mathbf{v} = -\nabla\Phi$

$$\Rightarrow \nabla\left(-\frac{\partial\Phi}{\partial t} + \tfrac{1}{2}v^2 + U + \frac{p}{\rho}\right) = 0$$

$$\Rightarrow -\frac{\partial\Phi}{\partial t} + \tfrac{1}{2}v^2 + U + \frac{p}{\rho} = \text{constant (within the fluid)}$$

For the same system, the continuity condition becomes

$$\nabla\cdot\mathbf{v} = -\nabla^2\Phi = 0$$



Within fluid: $0 \le z \le h + \zeta$

$$-\frac{\partial \Phi}{\partial t} + \tfrac{1}{2} v^2 + g(z - h) = \text{constant}$$ (We have absorbed p_0 in "constant")

$$-\nabla^2 \Phi = 0$$

At surface: $z = h + \zeta$ with $\zeta = \zeta(x, y, t)$

$$\frac{d\zeta}{dt} = \frac{\partial \zeta}{\partial t} + v_x \frac{\partial \zeta}{\partial x} + v_y \frac{\partial \zeta}{\partial y} \quad \text{where } v_{x,y} = v_{x,y}(x, y, h+\zeta, t)$$

Full equations:

Within fluid: $0 \le z \le h + \zeta$

$$-\frac{\partial \Phi}{\partial t} + \tfrac{1}{2} v^2 + g(z - h) = \text{constant}$$ (We have absorbed p_0 in "constant")

$$-\nabla^2 \Phi = 0$$

At surface: $z = h + \zeta$ with $\zeta = \zeta(x, y, t)$

$$\frac{d\zeta}{dt} = \frac{\partial \zeta}{\partial t} + v_x \frac{\partial \zeta}{\partial x} + v_y \frac{\partial \zeta}{\partial y} \quad \text{where } v_{x,y} = v_{x,y}(x, y, h+\zeta, t)$$

Linearized equations:

$$\text{For } \quad 0 \le z \le h + \zeta: \quad -\frac{\partial \Phi}{\partial t} + g(z - h) = 0 \qquad -\nabla^2 \Phi = 0$$

$$\text{At surface}: \quad z = h + \zeta \qquad \frac{d\zeta}{dt} = \frac{\partial \zeta}{\partial t} = v_z(x, y, h+\zeta, t)$$

$$-\frac{\partial \Phi(x, y, h+\zeta, t)}{\partial t} + g\zeta = 0$$

For simplicity, keep only linear terms and assume that horizontal variation is only along *x*:

$$\text{For} \quad 0 \le z \le h + \zeta: \qquad \nabla^2 \Phi = \left(\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial x^2} \right) \Phi(x, z, t) = 0$$

Consider and periodic waveform: $\Phi(x, z, t) = Z(z) \cos(k(x - ct))$

$$\Rightarrow \left(\frac{d^2}{dz^2} - k^2 \right) Z(z) = 0$$

Boundary condition at bottom of tank: $v_z(x, 0, t) = 0$

$$\Rightarrow \frac{dZ}{dz}(0) = 0 \qquad Z(z) = A\cosh(kz)$$

For simplicity, keep only linear terms and assume that horizontal variation is only along *x* – continued:

$$\text{At surface: } \quad z = h+\zeta \qquad \frac{\partial \zeta}{\partial t} = v_z(x, h+\zeta, t) = -\frac{\partial \Phi(x, h+\zeta, t)}{\partial z}$$

$$-\frac{\partial \Phi(x, h+\zeta, t)}{\partial t} + g\zeta = 0$$

$$-\frac{\partial^2 \Phi(x, h+\zeta, t)}{\partial t^2} + g\frac{\partial \zeta}{\partial t} = -\frac{\partial^2 \Phi(x, h+\zeta, t)}{\partial t^2} - g\frac{\partial \Phi(x, h+\zeta, t)}{\partial z} = 0$$

$$\text{For} \quad \Phi(x, (h+\zeta), t) = A\cosh(k(h+\zeta))\cos(k(x-ct))$$

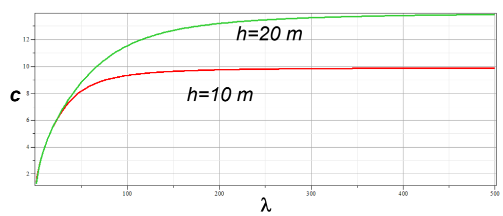
$$A\cosh(k(h+\zeta))\cos(k(x-ct))\left(k^2c^2 - gk\frac{\sinh(k(h+\zeta))}{\cosh(k(h+\zeta))}\right) = 0$$

$$\Rightarrow c^2 = \frac{g}{k}\frac{\sinh(k(h+\zeta))}{\cosh(k(h+\zeta))}$$

For simplicity, keep only linear terms and assume that horizontal variation is only along *x* – continued:

$$c^2 = \frac{g}{k}\frac{\sinh(k(h+\zeta))}{\cosh(k(h+\zeta))} = \frac{g}{k}\tanh(k(h+\zeta))$$

$$\text{Assuming } \zeta << h: \qquad c^2 = \frac{g}{k}\tanh(kh) \qquad \lambda = \frac{2\pi}{k}$$



For simplicity, keep only linear terms and assume that horizontal variation is only along *x* – continued:

$$c^2 \approx \frac{g}{k}\tanh(kh) \qquad \text{For } \lambda >> h,\ c^2 \approx gh$$

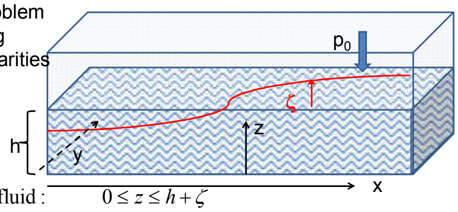
$$\Phi(x, z, t) = A\cosh(kz)\cos(k(x-ct))$$

$$\zeta(x,t) = \frac{1}{g}\frac{\partial \Phi(x, h+\zeta, t)}{\partial t} \approx \frac{kc}{g}A\cosh(kh)\sin(k(x-ct))$$

Note that for $\lambda >> h, \quad c^2 \approx gh$

(solutions are consistent with previous analysis)

General problem including non-linearities



Within fluid : $0 \le z \le h + \zeta$

$$-\frac{\partial \Phi}{\partial t} + \tfrac{1}{2}v^2 + g(z-h) = \text{constant}$$ (We have absorbed p_0 in our constant.)

$$-\nabla^2 \Phi = 0$$

At surface : $z = h + \zeta$ with $\zeta = \zeta(x, y, t)$

$$\frac{d\zeta}{dt} = \frac{\partial \zeta}{\partial t} + v_x \frac{\partial \zeta}{\partial x} + v_y \frac{\partial \zeta}{\partial y}$$ where $v_{x,y} = v_{x,y}(x, y, h+\zeta, t)$

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