

**PHY 711 Classical Mechanics and  
Mathematical Methods  
11-11:50 AM MWF Olin 107**

**Plan for Lecture 29:**

**Chapter 10 in F & W: Surface waves  
-- Non-linear contributions and soliton  
solutions**

11/09/2016

PHY 711 Fall 2016 -- Lecture 29

1

---

---

---

---

---

---

---

---

---

---

20	Mon, 10/17/2016	Chap. 5	Mechanics of rigid bodies	Exam due	
21	Wed, 10/19/2016	Chap. 5	Mechanics of rigid bodies	#16	10/24/2016
	Fri, 10/21/2016		Fall break -- no class		
22	Mon, 10/24/2016	Chap. 8	Mechanics of Elastic Membranes	#17	10/28/2016
23	Wed, 10/26/2016	Chap. 9	Introduction to hydrodynamics		
24	Fri, 10/28/2016	Chap. 9	Introduction to hydrodynamics	#18	10/31/2016
25	Mon, 10/31/2016	Chap. 9	Sound waves	#19	11/02/2016
26	Wed, 11/02/2016	Chap. 9	Sound waves	#20	11/04/2016
27	Fri, 11/04/2016	Chap. 9	Non-linear sound	#21	11/07/2016
28	Mon, 11/07/2016	Chap. 10	Surface waves in fluids		
29	Wed, 11/09/2016	Chap. 10	Surface waves in fluids	#22	11/11/2016
30	Fri, 11/11/2016				
31	Mon, 11/14/2016				
32	Wed, 11/16/2016				
33	Fri, 11/18/2016				
34	Mon, 11/21/2016				
	Wed, 11/23/2016		Thanksgiving Holiday -- no class		
	Fri, 11/25/2016		Thanksgiving Holiday -- no class		
	Wed, 12/07/2016		Presentations I		
	Fri, 12/09/2016		Presentations II		



11/09/2016

PHY 711 Fall 2016 -- Lecture 29

2

---

---

---

---

---

---

---


---

---


---

**Department of Physics**


**News**



**Congratulations to Dr. Maxim Zambaki, recent Ph.D. Recipient**



**Ryan Melvin Awarded Predoctoral Fellowship**



**Congratulations to Dr. Katerin Goetz, recent Ph.D. Recipient**

**Events**

**Wed. Nov. 9, 2016**  
**Science courses for non-science majors, they need us and we need them!**  
**Professor Shannon Willoughby, Montana State University**  
**4:00pm - Olin 101**  
**Refreshments served 3:30pm - Olin Lounge**

11/09/2016

PHY 711 Fall 2016 -- Lecture 29

3

---

---

---

---

---

---

---

---

---

---

1

**Surface waves in an incompressible fluid**

General problem including non-linearities

Within fluid:  $0 \leq z \leq h + \zeta$

$$-\frac{\partial \Phi}{\partial t} + \frac{1}{2}v^2 + g(z-h) = \text{constant} \quad \Phi = \Phi(x, y, z, t)$$

$$-\nabla^2 \Phi = 0 \quad \mathbf{v} = \mathbf{v}(x, y, z, t) = -\nabla \Phi(x, y, z, t)$$

At surface:  $z = h + \zeta$  with  $\zeta = \zeta(x, y, t)$

$$\frac{d\zeta}{dt} = \frac{\partial \zeta}{\partial t} + v_x \frac{\partial \zeta}{\partial x} + v_y \frac{\partial \zeta}{\partial y} = -\frac{\partial \Phi(x, y, z, t)}{\partial z} \Big|_{z=h+\zeta}$$

where  $v_{x,y} = v_{x,y}(x, y, h + \zeta, t)$

11/09/2016 PHY 711 Fall 2016 -- Lecture 29 4

---

---

---

---

---

---

---

---

---

---

Further simplifications; assume trivial y-dependence

$$\Phi = \Phi(x, z, t) \quad \zeta = \zeta(x, t)$$

Within fluid:  $0 \leq z \leq h + \zeta$

At surface:  $v_z(x, z = h + \zeta, t) = -\frac{\partial \Phi}{\partial z} = \frac{d\zeta}{dt}$

11/09/2016 PHY 711 Fall 2016 -- Lecture 29 5

---

---

---

---

---

---

---

---

---

---

Non-linear effects in surface waves:

Dominant non-linear effects  $\Rightarrow$  soliton solutions

$$\zeta(x, t) = \eta_0 \operatorname{sech}^2 \left( \sqrt{\frac{3\eta_0}{h}} \frac{x - ct}{2h} \right) \quad \eta_0 = \text{constant}$$

where  $c = \sqrt{\frac{gh}{1 - \eta_0/h}} \approx \sqrt{gh} \left( 1 + \frac{\eta_0}{2h} \right)$

11/09/2016 PHY 711 Fall 2016 -- Lecture 29 6

---

---

---

---

---

---

---

---

---

---

**Detailed analysis of non-linear surface waves**  
 [Note that these derivations follow Alexander L. Fetter and John Dirk Walecka, *Theoretical Mechanics of Particles and Continua* (McGraw Hill, 1980), Chapt. 10.]  
 We assume that we have an incompressible fluid:  $\rho = \text{constant}$   
 Velocity potential:  $\Phi(x, z, t)$ ;  $\mathbf{v}(x, z, t) = -\nabla\Phi(x, z, t)$

The surface of the fluid is described by  $z=h+\zeta(x, t)$ . It is assumed that the fluid is contained in a structure (lake, river, swimming pool, etc.) with a structureless bottom defined by the  $z = 0$  plane and filled to an equilibrium height of  $z = h$ .

11/09/2016 PHY 711 Fall 2016 -- Lecture 29 7

---

---

---

---

---

---

---

---

---

---

Defining equations for  $\Phi(x, z, t)$  and  $\zeta(x, t)$   
 where  $0 \leq z \leq h + \zeta(x, t)$   
 Continuity equation:  

$$\nabla \cdot \mathbf{v} = 0 \Rightarrow \frac{\partial^2 \Phi(x, z, t)}{\partial x^2} + \frac{\partial^2 \Phi(x, z, t)}{\partial z^2} = 0$$
 Bernoulli equation (assuming irrotational flow) and gravitation potential energy  

$$-\frac{\partial \Phi(x, z, t)}{\partial t} + \frac{1}{2} \left[ \underbrace{\left( \frac{\partial \Phi(x, z, t)}{\partial x} \right)^2}_{v_x^2} + \underbrace{\left( \frac{\partial \Phi(x, z, t)}{\partial z} \right)^2}_{v_z^2} \right] + g(z - h) = 0.$$

11/09/2016 PHY 711 Fall 2016 -- Lecture 29 8

---

---

---

---

---

---

---

---

---

---

Boundary conditions on functions –

Zero velocity at bottom of tank:  

$$\frac{\partial \Phi(x, 0, t)}{\partial z} = 0.$$

Consistent vertical velocity at water surface  

$$v_z(x, z, t)|_{z=h+\zeta} = \frac{d\zeta}{dt} = \mathbf{v} \cdot \nabla \zeta + \frac{\partial \zeta}{\partial t}$$

$$= v_x \frac{\partial \zeta}{\partial x} + \frac{\partial \zeta}{\partial t}$$

$$\Rightarrow -\frac{\partial \Phi(x, z, t)}{\partial z} + \frac{\partial \Phi(x, z, t)}{\partial x} \frac{\partial \zeta(x, t)}{\partial x} - \frac{\partial \zeta(x, t)}{\partial t} \Big|_{z=h+\zeta} = 0$$

11/09/2016 PHY 711 Fall 2016 -- Lecture 29 9

---

---

---

---

---

---

---

---

---

---




Coupled equations:  $-\frac{\partial \phi}{\partial t} + \frac{h^2}{2} \frac{\partial^3 \phi}{\partial t \partial x^2} + \frac{1}{2} \left( \frac{\partial \phi}{\partial x} \right)^2 + g\zeta = 0.$   
 $\frac{\partial}{\partial x} \left( (h + \zeta(x,t)) \frac{\partial \phi}{\partial x} \right) - \frac{h^3}{3!} \frac{\partial^4 \phi}{\partial x^4} - \frac{\partial \zeta}{\partial t} = 0.$

Traveling wave solutions with new notation:  
 $u \equiv x - ct \quad \phi(x,t) \equiv \chi(u) \quad \text{and} \quad \zeta(x,t) \equiv \eta(u)$

Note that the wave "speed"  $c$  will be consistently determined

$c \frac{d\chi(u)}{du} - \frac{ch^2}{2} \frac{d^3\chi(u)}{du^3} + \frac{1}{2} \left( \frac{d\chi(u)}{du} \right)^2 + g\eta(u) = 0.$   
 $\frac{d}{du} \left( (h + \eta(u)) \frac{d\chi(u)}{du} \right) - \frac{h^3}{6} \frac{d^4\chi(u)}{du^4} + c \frac{d\eta(u)}{du} = 0.$



11/09/2016 PHY 711 Fall 2016 -- Lecture 29 13

---

---

---

---

---

---

---

---

---

---

Integrating and re-arranging coupled equations

$c \frac{d\chi(u)}{du} - \frac{ch^2}{2} \frac{d^3\chi(u)}{du^3} + \frac{1}{2} \left( \frac{d\chi(u)}{du} \right)^2 + g\eta(u) = 0.$   
 $\Rightarrow \chi' = -\frac{g}{c}\eta + \frac{h^2}{2}\chi''' - \frac{1}{2c}(\chi')^2 \approx -\frac{g}{c}\eta - \frac{h^2g}{2c}\eta'' - \frac{g^2}{2c^3}\eta^2$

$\frac{d}{du} \left( (h + \eta(u)) \frac{d\chi(u)}{du} \right) - \frac{h^3}{6} \frac{d^4\chi(u)}{du^4} + c \frac{d\eta(u)}{du} = 0.$   
 $\Rightarrow (h + \eta)\chi' - \frac{h^3}{6}\chi'''' + c\eta' = 0$

11/09/2016 PHY 711 Fall 2016 -- Lecture 29 14

---

---

---

---

---

---

---

---

---

---

Integrating and re-arranging coupled equations – continued --  
 Expressing modified surface velocity equation in terms of  $\eta(u)$ :

$(h + \eta) \left( -\frac{g}{c}\eta - \frac{h^2g}{2c}\eta'' - \frac{g^2}{2c^3}\eta^2 \right) + \frac{h^3g}{6c}\eta'' + c\eta' = 0$   
 $\Rightarrow \left( 1 - \frac{gh}{c^2} \right) \eta - \frac{gh^3}{3c^2}\eta'' - \frac{g}{c^2} \left( 1 + \frac{gh}{2c^2} \right) \eta^2 = 0$   
 $\Rightarrow \left( 1 - \frac{hg}{c^2} \right) \eta(u) - \frac{h^2}{3}\eta''(u) - \frac{3}{2h}[\eta(u)]^2 = 0.$

Note:  $c^2 = gh + \dots$

11/09/2016 PHY 711 Fall 2016 -- Lecture 29 15

---

---

---

---

---

---

---

---

---

---

Solution of the famous Korteweg-de Vries equation

Modified surface amplitude equation in terms of  $\eta$

$$\Rightarrow \left(1 - \frac{hg}{c^2}\right)\eta(u) - \frac{h^2}{3}\eta''(u) - \frac{3}{2h}[\eta(u)]^2 = 0.$$

Soliton solution

$$\zeta(x,t) = \eta(x-ct) = \eta_0 \operatorname{sech}^2\left(\sqrt{\frac{3\eta_0}{h}} \frac{x-ct}{2h}\right)$$

$$c = \sqrt{\frac{gh}{1-\eta_0/h}} \approx \sqrt{gh} \left(1 + \frac{\eta_0}{2h}\right) \text{ where } \eta_0 \text{ is a constant}$$

11/09/2016 PHY 711 Fall 2016 -- Lecture 29 16

---

---

---

---

---

---

---

---


$$\zeta(x,t) = \eta(x-ct) = \eta_0 \operatorname{sech}^2\left(\sqrt{\frac{3\eta_0}{h}} \frac{x-ct}{2h}\right)$$

11/09/2016 PHY 711 Fall 2016 -- Lecture 29 17

---

---

---

---

---

---

---

---

Relationship to "standard" form of Korteweg-de Vries equation

New variables:

$$\beta = 2\eta_0, \quad \bar{x} = \sqrt{\frac{3}{2h}} \frac{x}{h}, \quad \text{and} \quad \bar{t} = \sqrt{\frac{3}{2h}} \frac{ct}{2\eta_0 h}.$$

Standard Korteweg-de Vries equation

$$\frac{\partial \eta}{\partial \bar{t}} + 6\eta \frac{\partial \eta}{\partial \bar{x}} + \frac{\partial^3 \eta}{\partial \bar{x}^3} = 0.$$

Soliton solution:

$$\eta(\bar{x}, \bar{t}) = \frac{\beta}{2} \operatorname{sech}^2\left[\frac{\sqrt{\beta}}{2}(\bar{x} - \beta \bar{t})\right].$$

11/09/2016 PHY 711 Fall 2016 -- Lecture 29 18

---

---

---

---

---

---

---

---

**More details**

Modified surface amplitude equation in terms of  $\eta$  :

$$\left(1 - \frac{hg}{c^2}\right)\eta(u) - \frac{h^2}{3}\eta''(u) - \frac{3}{2h}[\eta(u)]^2 = 0.$$

Some identities:  $\frac{\eta_0}{h} = 1 - \frac{gh}{c^2}$ ;  $\frac{\partial \eta}{\partial t} = -c \frac{d\eta}{du}$ ;  $\frac{\partial \eta}{\partial x} = \frac{d\eta}{du}$ .

Derivative of surface amplitude equation:

$$\frac{\eta_0}{h}\eta' - \frac{h^2}{3}\eta''' - \frac{3}{h}\eta\eta' = 0.$$

Expression in terms of  $x$  and  $t$ :

$$-\frac{\eta_0}{ch}\frac{\partial \eta}{\partial t} - \frac{h^2}{3}\frac{\partial^3 \eta}{\partial x^3} - \frac{3}{h}\eta\frac{\partial \eta}{\partial x} = 0.$$

Expression in terms of  $\bar{x}$  and  $\bar{t}$ :

$$\frac{\partial \eta}{\partial \bar{t}} + 6\eta\frac{\partial \eta}{\partial \bar{x}} + \frac{\partial^3 \eta}{\partial \bar{x}^3} = 0.$$

11/09/2016 PHY 711 Fall 2016 -- Lecture 29 19

---

---

---

---

---

---

---

---

---

---

Steps to solution

$$\left(1 - \frac{hg}{c^2}\right)\eta(u) - \frac{h^2}{3}\eta''(u) - \frac{3}{2h}[\eta(u)]^2 = 0.$$

Let  $1 - \frac{hg}{c^2} \equiv \frac{\eta_0}{h} \Rightarrow \frac{\eta_0}{h}\eta(u) - \frac{h^2}{3}\eta''(u) - \frac{3}{2h}[\eta(u)]^2 = 0.$

Multiply equation by  $\eta'(u) \Rightarrow \frac{d}{du}\left(\frac{\eta_0}{2h}\eta^2(u) - \frac{h^2}{6}\eta'^2(u) - \frac{1}{2h}\eta^3(u)\right) = 0$

Integrate wrt  $u$  and assume solution vanishes for  $u \rightarrow \infty$

$$\frac{\eta_0}{2h}\eta^2(u) - \frac{h^2}{6}\eta'^2(u) - \frac{1}{2h}\eta^3(u) = 0$$

$$\eta'^2(u) = \frac{3}{h^3}\eta^2(u)(\eta_0 - \eta(u))$$

$$\frac{d\eta}{\eta(\eta_0 - \eta)^{1/2}} = \sqrt{\frac{3}{h^3}} du \Rightarrow \eta(u) = \frac{\eta_0}{\cosh^2\left(\sqrt{\frac{3\eta_0}{4h^3}}u\right)}$$

11/09/2016 PHY 711 Fall 2016 -- Lecture 29 20

---

---

---

---

---

---

---

---

---

---

**Summary**

Soliton solution

$$\zeta(x,t) = \eta(x-ct) = \eta_0 \operatorname{sech}^2\left(\sqrt{\frac{3\eta_0}{h}}\frac{x-ct}{2h}\right)$$

$$c = \sqrt{\frac{gh}{1-\eta_0/h}} \approx \sqrt{gh}\left(1 + \frac{\eta_0}{2h}\right) \text{ where } \eta_0 \text{ is a constant}$$

11/09/2016 PHY 711 Fall 2016 -- Lecture 29 21

---

---

---

---

---

---

---

---

---

---

Photo of canal soliton <http://www.ma.hw.ac.uk/solitons/>



11/09/2016

PHY 711 Fall 2016 -- Lecture 29

22

---

---

---

---

---

---

---

---