PHY 711 Classical Mechanics and Mathematical Methods
11-11:50 AM MWF Olin 107

Plan for Lecture 30:

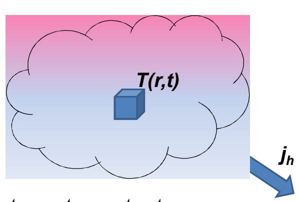
Chapter 11 in F & W:

Heat conduction

1. **Basic equations**
2. **Boundary value problems**

20	Mon, 10/17/2016	Chap. 5	Mechanics of rigid bodies	Exam due	
21	Wed, 10/19/2016	Chap. 5	Mechanics of rigid bodies	#16	10/24/2016
	Fri, 10/21/2016		Fall break -- no class		
22	Mon, 10/24/2016	Chap. 8	Mechanics of Elastic Membranes	#17	10/28/2016
23	Wed, 10/26/2016	Chap. 9	Introduction to hydrodynamics		
24	Fri, 10/28/2016	Chap. 9	Introduction to hydrodynamics	#18	10/31/2016
25	Mon, 10/31/2016	Chap. 9	Sound waves	#19	11/02/2016
26	Wed, 11/02/2016	Chap. 9	Sound waves	#20	11/04/2016
27	Fri, 11/04/2016	Chap. 9	Non-linear sound	#21	11/07/2016
28	Mon, 11/07/2016	Chap. 10	Surface waves in fluids		
29	Wed, 11/09/2016	Chap. 10	Surface waves in fluids	#22	11/11/2016
30	Fri, 11/11/2016	Chap. 11	Heat conductivity	#23	11/14/2016
31	Mon, 11/14/2016				
32	Wed, 11/16/2016				
33	Fri, 11/18/2016				
34	Mon, 11/21/2016				
	Wed, 11/23/2016		Thanksgiving Holiday -- no class		
	Fri, 11/25/2016		Thanksgiving Holiday -- no class		
	Wed, 12/07/2016		Presentations I		
	Fri, 12/09/2016		Presentations II		

Conduction of heat



Enthalpy of a system at constant pressure p

non uniform temperature $T(\mathbf{r},t)$

mass density ρ and heat capacity c_p

$$H = \int_V \rho c_p \left(T(\mathbf{r},t) - T_0\right) d^3r + H_0(T_0, p)$$

Note that in this treatment we are considering a system at constant pressure p

Notation:
- Heat added to system -- $dQ = TdS$
- External work done on system -- $dW = -pdV$
- Internal energy -- $dE = dQ + dW = TdS - pdV$
- Entropy -- dS
- Enthalpy -- $dH = d(E + pV) = TdS + Vdp$

Heat capacity at constant pressure:

$$C_p \equiv \left(\frac{dQ}{dT}\right)_p = \left(\frac{\partial H}{\partial T}\right)_p = T\left(\frac{\partial S}{\partial T}\right)_p$$

$$C_p = \rho c_p d^3r$$

More generally, note that c_p can depend on T; we are assume that dependence to be trivial.

Conduction of heat -- continued

$$H = \int_V \rho c_p \left(T(\mathbf{r},t) - T_0\right) d^3r + H_0(T_0, p)$$

Time rate of change of enthapy:

$$\frac{dH}{dt} = \int_V \rho c_p \frac{\partial T(\mathbf{r},t)}{\partial t} d^3r = -\int_A \mathbf{j}_h \cdot d\mathbf{A} + \int_V \rho \dot{q} d^3r$$

heat flux

heat source

$$\rho c_p \frac{\partial T(\mathbf{r},t)}{\partial t} = -\nabla \cdot \mathbf{j}_h + \rho \dot{q}$$

Conduction of heat -- continued

$$\rho c_p \frac{\partial T(\mathbf{r},t)}{\partial t} = -\nabla \cdot \mathbf{j}_h + \rho \dot{q}$$

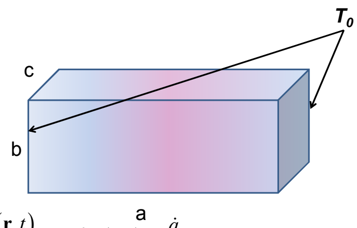
Empirically: $\mathbf{j}_h = -k_{th} \nabla T(\mathbf{r},t)$

$$\Rightarrow \frac{\partial T(\mathbf{r},t)}{\partial t} = \kappa \nabla^2 T(\mathbf{r},t) + \frac{\dot{q}}{c_p}$$

$\kappa \equiv \frac{k_{th}}{\rho c_p}$ thermal diffusivity

	Typical values (m^2/s)
Air	2×10^{-5}
Water	1×10^{-7}
Copper	1×10^{-4}

Boundary value problems for heat conduction

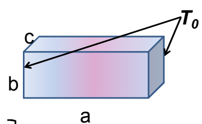


$$\frac{\partial T(\mathbf{r},t)}{\partial t} - \kappa\nabla^2 T(\mathbf{r},t) = \frac{\dot{q}}{c_p}$$

Without source term: $\frac{\partial T(\mathbf{r},t)}{\partial t} - \kappa\nabla^2 T(\mathbf{r},t) = 0$

Example with boundary values: $T(0,y,z,t) = T(a,y,z,t) = T_0$

Boundary value problems for heat conduction



$$\frac{\partial T(\mathbf{r},t)}{\partial t} - \kappa\nabla^2 T(\mathbf{r},t) = 0$$

$$T(0,y,z,t) = T(a,y,z,t) = T_0$$

$$\frac{\partial T(x,0,z,t)}{\partial y} = \frac{\partial T(x,b,z,t)}{\partial y} = 0$$

$$\frac{\partial T(x,y,0,t)}{\partial z} = \frac{\partial T(x,y,c,t)}{\partial z} = 0$$

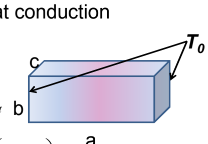
Assuming thermally insulated boundaries

Separation of varables : $T(x,y,z,t) = T_0 + X(x)Y(y)Z(z)e^{-\lambda t}$

Let $\frac{d^2X}{dx^2} = -\alpha^2 X \quad \frac{d^2Y}{dy^2} = -\beta^2 Y \quad \frac{d^2Z}{dz^2} = -\gamma^2 Z$

$$\Rightarrow -\lambda + \kappa(\alpha^2 + \beta^2 + \gamma^2) = 0$$

Boundary value problems for heat conduction



$$T(x,y,z,t) = T_0 + X(x)Y(y)Z(z)e^{-\lambda t}$$

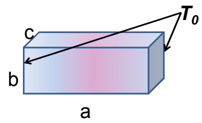
$$X(0) = X(a) = 0 \qquad \Rightarrow X(x) = \sin\left(\frac{m\pi x}{a}\right)$$

$$\frac{dY(0)}{dy} = \frac{dY(b)}{dy} = 0 \qquad \Rightarrow Y(y) = \cos\left(\frac{n\pi y}{b}\right)$$

$$\frac{dZ(0)}{dz} = \frac{dZ(c)}{dz} = 0 \qquad \Rightarrow Z(z) = \cos\left(\frac{p\pi z}{c}\right)$$

$$-\lambda_{nmp} + \kappa\left(\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 + \left(\frac{p\pi}{c}\right)^2\right) = 0$$

Boundary value problems for heat conduction



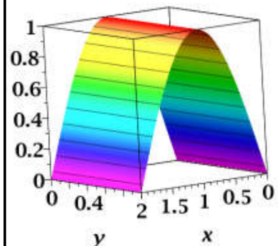
Full solution:

$$T(x,y,z,t) = T_0 + \sum_{nmp} C_{nmp} \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\cos\left(\frac{p\pi z}{c}\right) e^{-\lambda_{mnp} t}$$

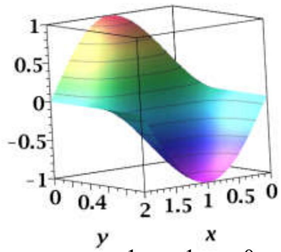
$$\lambda_{nmp} = \kappa\left(\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 + \left(\frac{p\pi}{c}\right)^2\right)$$

Full solution:

$$T(x,y,z,t) = T_0 + \sum_{nmp} C_{nmp} \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\cos\left(\frac{p\pi z}{c}\right) e^{-\lambda_{mnp} t}$$



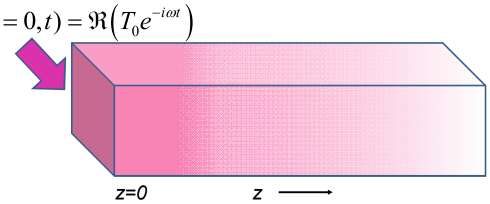
$n = 1, m = 0, p = 0$



$n = 1, m = 1, p = 0$

Oscillatory thermal behavior

$$T(z=0,t) = \Re\left(T_0 e^{-i\omega t}\right)$$



$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial z^2}$$

Assume: $T(z,t) = \Re\left(f(z) e^{-i\omega t}\right)$

$$(-i\omega) f = \kappa \frac{d^2 f}{dz^2}$$

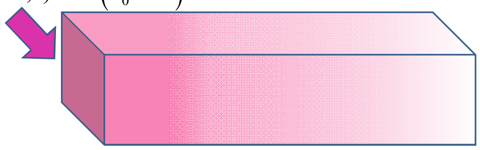
Let $f(z) = Ae^{\alpha z}$

$$\alpha^2 = -\frac{i\omega}{\kappa} = e^{3i\pi/2}\ \frac{\omega}{\kappa}$$

$$\alpha = \pm(1-i)\sqrt{\frac{\omega}{2\kappa}}$$

Oscillatory thermal behavior -- continued

$$T(z=0,t)=\Re\left(T_0 e^{-i\omega t}\right)$$

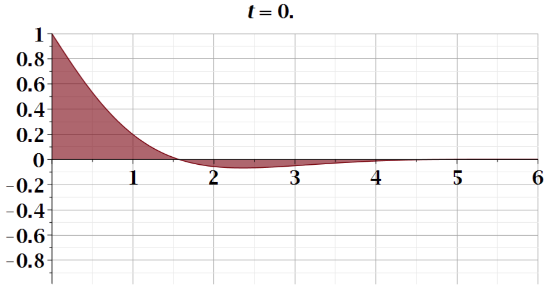


$$T(z,t)=\Re\left(Ae^{\pm(1-i)z/\delta}e^{-i\omega t}\right)$$

where $\delta \equiv \sqrt{\dfrac{2\kappa}{\omega}}$

Physical solution: $T(z,t)=T_0 e^{-z/\delta}\cos\left(\dfrac{z}{\delta}-\omega t\right)$

$$T(z,t)=T_0 e^{-z/\delta}\cos\left(\frac{z}{\delta}-\omega t\right)$$



Initial value problem in an infinite domain; Fourier transform

$$\frac{\partial T(\mathbf{r},t)}{\partial t}-\kappa\nabla^2 T(\mathbf{r},t)=0$$

$$T(\mathbf{r},0)=f(\mathbf{r})$$

Let : $\tilde{T}(\mathbf{q},t)=\int d^3r\, e^{-i\mathbf{q}\cdot\mathbf{r}}T(\mathbf{r},t)$

$$\tilde{f}(\mathbf{q})=\int d^3r\, e^{-i\mathbf{q}\cdot\mathbf{r}}f(\mathbf{r})$$

$$\Rightarrow \tilde{T}(\mathbf{q},0)=\tilde{f}(\mathbf{q})$$

$$\Rightarrow \frac{\partial \tilde{T}(\mathbf{q},t)}{\partial t}=-\kappa q^2\tilde{T}(\mathbf{q},t)$$

$$\tilde{T}(\mathbf{q},t)=\tilde{T}(\mathbf{q},0)e^{-\kappa q^2 t}$$

Initial value problem in an infinite domain; Fourier transform

$$\widetilde{T}(\mathbf{q},t) = \int d^3 r e^{-i\mathbf{q}\cdot\mathbf{r}} T(\mathbf{r},t) \qquad \Rightarrow T(\mathbf{r},t) = \frac{1}{(2\pi)^3}\int d^3 q e^{i\mathbf{q}\cdot\mathbf{r}} \widetilde{T}(\mathbf{q},t)$$

$$\widetilde{T}(\mathbf{q},t) = \widetilde{T}(\mathbf{q},0) e^{-\kappa q^2 t}$$

$$T(\mathbf{r},t) = \frac{1}{(2\pi)^3}\int d^3 q e^{i\mathbf{q}\cdot\mathbf{r}} \widetilde{T}(\mathbf{q},0) e^{-\kappa q^2 t}$$

$$\widetilde{T}(\mathbf{q},0) = \widetilde{f}(\mathbf{q}) = \int d^3 r e^{-i\mathbf{q}\cdot\mathbf{r}} f(\mathbf{r})$$

$$T(\mathbf{r},t) = \int d^3 r' G(\mathbf{r}-\mathbf{r}',t) T(\mathbf{r}',0)$$

$$\text{with } G(\mathbf{r}-\mathbf{r}',t) \equiv \frac{1}{(2\pi)^3}\int d^3 q e^{i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')} e^{-\kappa q^2 t}$$

Initial value problem in an infinite domain; Fourier transform

$$T(\mathbf{r},t) = \int d^3 r' G(\mathbf{r}-\mathbf{r}',t) T(\mathbf{r}',0)$$

$$\text{with } G(\mathbf{r}-\mathbf{r}',t) \equiv \frac{1}{(2\pi)^3}\int d^3 q e^{i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')} e^{-\kappa q^2 t}$$

$$G(\mathbf{r}-\mathbf{r}',t) = \frac{1}{(4\pi\kappa t)^{3/2}} e^{-|\mathbf{r}-\mathbf{r}'|^2/(4\kappa t)}$$

Heat equation in half-space

$$\frac{\partial T(\mathbf{r},t)}{\partial t} - \kappa \nabla^2 T(\mathbf{r},t) = 0$$

$$T(\mathbf{r},t) \Rightarrow T(z,t) \quad \text{with initial and boundary values:}$$

$$T(z,t) \equiv 0 \quad \text{for } z < 0$$

$$T(z,0) = 0 \quad \text{for } z > 0$$

$$T(0,t) = T_0 \quad \text{for } t \geq 0$$

$$\text{Solution:} \quad T = T_0 \,\mathrm{erfc}\left(\frac{z}{2\sqrt{\kappa t}}\right)$$

$$\text{where } \mathrm{erfc}(x) \equiv \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2} du$$

Heat equation in half-space -- continued

$$\frac{\partial T(z,t)}{\partial t} - \kappa \frac{\partial^2 T(z,t)}{\partial z^2} = 0$$

Solution : $T = T_0 \,\mathrm{erfc}\left(\frac{z}{2\sqrt{\kappa t}}\right)$

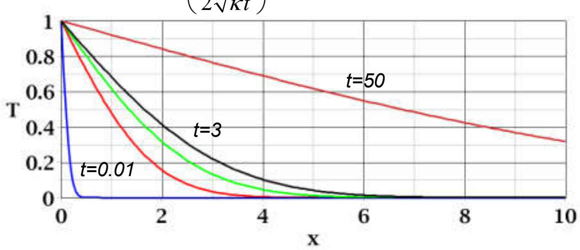
where $\mathrm{erfc}(x) \equiv \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2} du$

Note that $\frac{d\,\mathrm{erfc}(x)}{dx} = \frac{d}{dx} \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2} du = -\frac{2}{\sqrt{\pi}} e^{-x^2}$

$$\frac{\partial}{\partial t} \mathrm{erfc}\left(\frac{z}{2\sqrt{\kappa t}}\right) = \frac{2}{\sqrt{\pi}} e^{-\left(z^2/(4\kappa t)\right)} \left(\frac{z}{4\sqrt{\kappa t^3}}\right)$$

$$\frac{\partial^2}{\partial z^2} \mathrm{erfc}\left(\frac{z}{2\sqrt{\kappa t}}\right) = \frac{2}{\sqrt{\pi}} e^{-\left(z^2/(4\kappa t)\right)} \left(\frac{z}{4\kappa\sqrt{\kappa t^3}}\right)$$

$$T = T_0 \,\mathrm{erfc}\left(\frac{z}{2\sqrt{\kappa t}}\right)$$



Temperature profile