

**PHY 711 Classical Mechanics and  
Mathematical Methods  
11-11:50 AM MWF Olin 107**

**Plan for Lecture 31**

**Viscous fluids – Chap. 12 in F & W**

- 1. Viscous stress tensor**
- 2. Navier-Stokes equation**
- 3. Example for incompressible fluid**

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21 Wed, 10/19/2016	Chap. 5	Mechanics of rigid bodies	#16	10/24/2016
Fri, 10/21/2016		Fall break – no class		
22 Mon, 10/24/2016	Chap. 8	Mechanics of Elastic Membranes	#17	10/28/2016
23 Wed, 10/26/2016	Chap. 9	Introduction to hydrodynamics		
24 Fri, 10/28/2016	Chap. 9	Introduction to hydrodynamics	#18	10/31/2016
25 Mon, 10/31/2016	Chap. 9	Sound waves	#19	11/02/2016
26 Wed, 11/02/2016	Chap. 9	Sound waves	#20	11/04/2016
27 Fri, 11/04/2016	Chap. 9	Non-linear sound	#21	11/07/2016
28 Mon, 11/07/2016	Chap. 10	Surface waves in fluids		
29 Wed, 11/09/2016	Chap. 10	Surface waves in fluids	#22	11/11/2016
30 Fri, 11/11/2016	Chap. 11	Heat conductivity	#23	11/14/2016
31 Mon, 11/14/2016	Chap. 12	Viscous fluids	#24	11/16/2016
32 Wed, 11/16/2016				
33 Fri, 11/18/2016				
34 Mon, 11/21/2016				
Wed, 11/23/2016		Thanksgiving Holiday – no class		
Fri, 11/25/2016		Thanksgiving Holiday – no class		
Wed, 12/07/2016		Presentations I		
Fri, 12/09/2016		Presentations II		

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**Equations for motion of non-viscous fluid**

Newton-Euler equation of motion:

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho(\mathbf{v} \cdot \nabla) \mathbf{v} = \rho \mathbf{f}_{\text{applied}} - \nabla p$$

Continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \quad \Rightarrow \quad \mathbf{v} \cdot \left( \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) \right) = 0$$

Add two equations:

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \frac{\partial \rho}{\partial t} \mathbf{v} + \rho(\mathbf{v} \cdot \nabla) \mathbf{v} + \mathbf{v} \nabla \cdot (\rho \mathbf{v}) = \rho \mathbf{f}_{\text{applied}} - \nabla p$$

$$\frac{\partial(\rho \mathbf{v})}{\partial t} + \sum_{j=1}^3 \frac{\partial(\rho v_j \mathbf{v})}{\partial x_j} = \rho \mathbf{f}_{\text{applied}} - \nabla p$$

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Equations for motion of non-viscous fluid -- continued

Newton-Euler equation in terms of momentum:

$$\frac{\partial(\rho \mathbf{v})}{\partial t} + \sum_{j=1}^3 \frac{\partial(\rho v_j \mathbf{v})}{\partial x_j} = \rho \mathbf{f}_{\text{applied}} - \nabla p$$

$$\frac{\partial(\rho \mathbf{v})}{\partial t} + \sum_{j=1}^3 \frac{\partial(\rho v_j \mathbf{v})}{\partial x_j} + \nabla p = \rho \mathbf{f}_{\text{applied}}$$

Fluid momentum:  $\rho \mathbf{v}$

Stress tensor:  $T_{ij} \equiv \rho v_i v_j + p \delta_{ij}$

$i^{\text{th}}$  component of momentum form of Newton-Euler equation:

$$\frac{\partial(\rho v_i)}{\partial t} + \sum_{j=1}^3 \frac{\partial T_{ij}}{\partial x_j} = \rho f_i$$

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Now consider the effects of viscosity

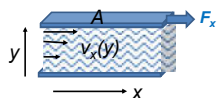
In terms of stress tensor:

$$T_{ij} = T_{ij}^{\text{ideal}} + T_{ij}^{\text{viscous}}$$

$$T_{ij}^{\text{ideal}} = \rho v_i v_j + p \delta_{ij} = T_{ji}^{\text{ideal}}$$

Newton's "law" of viscosity

$$\frac{F_x}{A} = \eta \frac{\partial v_x}{\partial y}$$



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Effects of viscosity

Argue that viscosity is due to shear forces in a fluid of the form:

$$\frac{F_{\text{drag}}}{A} = \eta \frac{\partial v_x}{\partial y}$$

Formulate viscosity stress tensor with traceless and diagonal terms:

$$T_{kl}^{\text{viscous}} = -\eta \left( \frac{\partial v_k}{\partial x_l} + \frac{\partial v_l}{\partial x_k} - \frac{2}{3} \delta_{kl} (\nabla \cdot \mathbf{v}) \right) - \zeta \delta_{kl} (\nabla \cdot \mathbf{v})$$

↑  
viscosity

↑  
bulk viscosity

Total stress tensor:  $T_{kl} = T_{kl}^{\text{ideal}} + T_{kl}^{\text{viscous}}$

$$T_{kl}^{\text{ideal}} = \rho v_k v_l + p \delta_{kl}$$

$$T_{kl}^{\text{viscous}} = -\eta \left( \frac{\partial v_k}{\partial x_l} + \frac{\partial v_l}{\partial x_k} - \frac{2}{3} \delta_{kl} (\nabla \cdot \mathbf{v}) \right) - \zeta \delta_{kl} (\nabla \cdot \mathbf{v})$$

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Example – steady flow of an incompressible fluid in a long pipe with a circular cross section of radius  $R$  -- continued

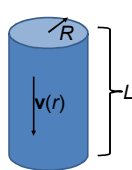
Navier-Stokes equation becomes:

$$0 = -\frac{1}{\rho} \nabla p + \frac{\eta}{\rho} \nabla^2 \mathbf{v}$$

Assume that  $\mathbf{v}(\mathbf{r}, t) = v_z(r) \hat{\mathbf{z}}$

$$\frac{\partial p}{\partial z} = \eta \nabla^2 v_z(r) \quad (\text{independent of } z)$$

Suppose that  $\frac{\partial p}{\partial z} = -\frac{\Delta p}{L}$  (uniform pressure gradient)

$$\Rightarrow \nabla^2 v_z(r) = -\frac{\Delta p}{\eta L}$$


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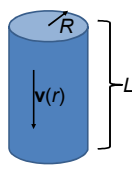
Example – steady flow of an incompressible fluid in a long pipe with a circular cross section of radius  $R$  -- continued

$$\nabla^2 v_z(r) = -\frac{\Delta p}{\eta L}$$

$$\frac{1}{r} \frac{d}{dr} r \frac{dv_z(r)}{dr} = -\frac{\Delta p}{\eta L}$$

$$v_z(r) = -\frac{\Delta p r^2}{4\eta L} + C_1 \ln(r) + C_2$$

$\Rightarrow C_1 = 0 \quad v_z(R) = 0 = -\frac{\Delta p R^2}{4\eta L} + C_2$

$$v_z(r) = \frac{\Delta p}{4\eta L} (R^2 - r^2)$$


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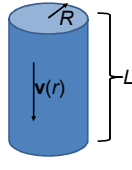
Example – steady flow of an incompressible fluid in a long pipe with a circular cross section of radius  $R$  -- continued

$$v_z(r) = \frac{\Delta p}{4\eta L} (R^2 - r^2)$$

Mass flow rate through the pipe:

$$\frac{dM}{dt} = 2\pi\rho \int_0^R r dr v_z(r) = \frac{\Delta p \rho \pi R^4}{8\eta L}$$

Poiseuille formula;  
 → Method for measuring  $\eta$



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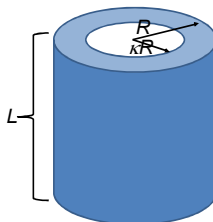
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Example – steady flow of an incompressible fluid in a long tube with a circular cross section of outer radius  $R$  and inner radius  $\kappa R$



$$\nabla^2 v_z(r) = -\frac{\Delta p}{\eta L}$$

$$\frac{1}{r} \frac{d}{dr} r \frac{dv_z(r)}{dr} = -\frac{\Delta p}{\eta L}$$

$$v_z(r) = -\frac{\Delta p r^2}{4\eta L} + C_1 \ln(r) + C_2$$

$$v_z(R) = 0 = -\frac{\Delta p R^2}{4\eta L} + C_1 \ln(R) + C_2$$

$$v_z(\kappa R) = 0 = -\frac{\Delta p \kappa^2 R^2}{4\eta L} + C_1 \ln(\kappa R) + C_2$$

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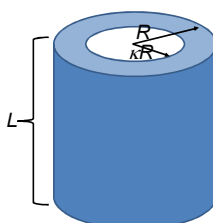
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Example – steady flow of an incompressible fluid in a long tube with a circular cross section of outer radius  $R$  and inner radius  $\kappa R$  -- continued



Solving for  $C_1$  and  $C_2$  :

$$v_z(r) = \frac{\Delta p R^2}{4\eta L} \left( 1 - \left(\frac{r}{R}\right)^2 - \frac{1 - \kappa^2}{\ln \kappa} \ln\left(\frac{r}{R}\right) \right)$$

Mass flow rate through the pipe:

$$\frac{dM}{dt} = 2\pi\rho \int_{\kappa R}^R r dv_z(r) = \frac{\Delta p \rho \pi R^4}{8\eta L} \left( 1 - \kappa^4 + \frac{(1 - \kappa^2)^2}{\ln \kappa} \right)$$

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