PHY 711 Classical Mechanics and Mathematical Methods
11-11:50 AM MWF Olin 107

Plan for Lecture 33

Navier-Stokes equation – Chap. 12 in F&W

- **Effects of viscosity on sound**

23	Wed, 10/26/2016	Chap. 9	Introduction to hydrodynamics		
24	Fri, 10/28/2016	Chap. 9	Introduction to hydrodynamics	#18	10/31/2016
25	Mon, 10/31/2016	Chap. 9	Sound waves	#19	11/02/2016
26	Wed, 11/02/2016	Chap. 9	Sound waves	#20	11/04/2016
27	Fri, 11/04/2016	Chap. 9	Non-linear sound	#21	11/07/2016
28	Mon, 11/07/2016	Chap. 10	Surface waves in fluids		
29	Wed, 11/09/2016	Chap. 10	Surface waves in fluids	#22	11/11/2016
30	Fri, 11/11/2016	Chap. 11	Heat conductivity	#23	11/14/2016
31	Mon, 11/14/2016	Chap. 12	Viscous fluids	#24	11/16/2016
32	Wed, 11/16/2016	Chap. 12	Viscous fluids	#25	11/18/2016
33	Fri, 11/18/2016	Chap. 12	Viscous fluids	#26	11/21/2016
34	Mon, 11/21/2016				
	Wed, 11/23/2016		Thanksgiving Holiday -- no class		
	Fri, 11/25/2016		Thanksgiving Holiday -- no class		
35	Mon, 11/28/2016				
36	Wed, 11/30/2016				
37	Fri, 12/02/2016				
38	Mon, 12/05/2016				
	Wed, 12/07/2016		Presentations I		
	Fri, 12/09/2016		Presentations II		

Comment on drag on a sphere moving in a viscous incompressible fluid:

$$v_r = u\cos\theta\left(1 - \frac{2}{r}\frac{df}{dr}\right) = u\cos\theta\left(1 - 4C_1 - \frac{2C_2}{r} - \frac{2C_4}{r^3}\right)$$

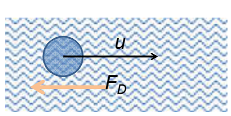
$$v_\theta = -u\sin\theta\left(1 - \frac{d^2 f}{dr^2} - \frac{1}{r}\frac{df}{dr}\right) = -u\sin\theta\left(1 - 4C_1 - \frac{C_2}{r} + \frac{C_4}{r^3}\right)$$

To satisfy $\mathbf{v}(r \to \infty) = \mathbf{u}$: $\quad \Rightarrow C_1 = 0$

To satisfy $\mathbf{v}(R) = 0$ solve for C_2, C_4 ← "non-slip" condition

$$v_r = u\cos\theta\left(1 - \frac{3R}{2r} + \frac{R^3}{2r^3}\right)$$

$$v_\theta = -u\sin\theta\left(1 - \frac{3R}{4r} - \frac{R^3}{4r^3}\right)$$



Newton-Euler equations for viscous fluids

Navier-Stokes equation

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{f} - \frac{1}{\rho}\nabla p + \frac{\eta}{\rho}\nabla^2 \mathbf{v} + \frac{1}{\rho}\left(\zeta + \frac{1}{3}\eta\right)\nabla(\nabla \cdot \mathbf{v})$$

Continuity condition

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

Newton-Euler equations for viscous fluids – effects on sound

Without viscosity terms:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{f} - \frac{1}{\rho}\nabla p \qquad \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

Assume: $\mathbf{v} = 0 + \delta\mathbf{v}$ $\quad \mathbf{f}=0$ $\quad \rho = \rho_0 + \delta\rho$

$$p = p_0 + \delta p = p_0 + \left(\frac{\partial p}{\partial \rho}\right)_s \delta\rho \equiv p_0 + c^2 \delta\rho$$

Linearized equations: $\frac{\partial \delta\mathbf{v}}{\partial t} = -\frac{c^2}{\rho_0}\nabla \delta\rho \qquad \frac{\partial \delta\rho}{\partial t} + \rho_0 \nabla \cdot (\delta\mathbf{v}) = 0$

Let $\delta\mathbf{v} \equiv \delta\mathbf{v}_0 \, e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \qquad \delta\rho \equiv \delta\rho_0 \, e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$

Sound waves without viscosity -- continued

Linearized equations: $\frac{\partial \delta\mathbf{v}}{\partial t} = -\frac{c^2}{\rho_0}\nabla \delta\rho \qquad \frac{\partial \delta\rho}{\partial t} + \rho_0 \nabla \cdot (\delta\mathbf{v}) = 0$

Let $\delta\mathbf{v} \equiv \delta\mathbf{v}_0 \, e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \qquad \delta\rho \equiv \delta\rho_0 \, e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$

$$\frac{\partial \delta\mathbf{v}}{\partial t} = -\frac{c^2}{\rho_0}\nabla \delta\rho \qquad \Rightarrow \omega \delta\mathbf{v}_0 = \frac{c^2 \delta\rho_0}{\rho_0}\mathbf{k}$$

$$\frac{\partial \delta\rho}{\partial t} + \rho_0 \nabla \cdot (\delta\mathbf{v}) = 0 \qquad \Rightarrow -\omega\delta\rho_0 + \rho_0 \mathbf{k} \cdot \delta\mathbf{v}_0 = 0$$

$$\Rightarrow k^2 = \frac{\omega^2}{c^2} \qquad \frac{\delta\rho_0}{\rho_0} = \frac{\hat{\mathbf{k}} \cdot \delta\mathbf{v}_0}{c}$$

➔Pure longitudinal harmonic wave solutions

Newton-Euler equations for viscous fluids – effects on sound
Recall full equations:

Navier-Stokes equation

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{f} - \frac{1}{\rho}\nabla p + \frac{\eta}{\rho}\nabla^2\mathbf{v} + \frac{1}{\rho}\left(\zeta + \frac{1}{3}\eta\right)\nabla(\nabla\cdot\mathbf{v})$$

Continuity condition

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0$$

Assume: $\mathbf{v} = 0 + \delta\mathbf{v}$ $\quad \mathbf{f}=0 \quad$ $\rho = \rho_0 + \delta\rho$

$$p(s,\rho) = p_0 + \delta p = p_0 + c^2\delta\rho + \left(\frac{\partial p}{\partial s}\right)_\rho \delta s$$

where $c^2 \equiv \left(\frac{\partial p}{\partial \rho}\right)_s$

viscosity causes heat transfer

Newton-Euler equations for viscous fluids – effects on sound
Note that pressure now depends both on density and entropy so that entropy must be coupled into the equations

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{f} - \frac{1}{\rho}\nabla p + \frac{\eta}{\rho}\nabla^2\mathbf{v} + \frac{1}{\rho}\left(\zeta + \frac{1}{3}\eta\right)\nabla(\nabla\cdot\mathbf{v})$$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0 \qquad \rho T\frac{\partial s}{\partial t} = k_{th}\nabla^2 T$$

Assume: $\mathbf{v} = 0 + \delta\mathbf{v}$ $\quad \mathbf{f}=0 \quad$ $\rho = \rho_0 + \delta\rho$

$$p = p_0 + \delta p = p_0 + c^2\delta\rho + \left(\frac{\partial p}{\partial s}\right)_\rho \delta s \quad \text{where } c^2 \equiv \left(\frac{\partial p}{\partial \rho}\right)_s$$

$$T = T_0 + \delta T = T_0 + \left(\frac{\partial T}{\partial \rho}\right)_s \delta\rho + \left(\frac{\partial T}{\partial s}\right)_\rho \delta s$$

$$s = s_0 + \delta s$$

Newton-Euler equations for viscous fluids –
linearized equations

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{f} - \frac{1}{\rho}\nabla p + \frac{\eta}{\rho}\nabla^2\mathbf{v} + \frac{1}{\rho}\left(\zeta + \frac{1}{3}\eta\right)\nabla(\nabla\cdot\mathbf{v})$$

$$\Rightarrow \frac{\partial \delta\mathbf{v}}{\partial t} = -\underbrace{\frac{1}{\rho_0}\nabla\delta p}_{} + \frac{\eta}{\rho}\nabla^2\delta\mathbf{v} + \frac{1}{\rho_0}\left(\zeta + \frac{1}{3}\eta\right)\nabla(\nabla\cdot\delta\mathbf{v})$$

$$-\frac{1}{\rho_0}\left\{\left(\frac{\partial p}{\partial \rho}\right)_s \nabla\delta\rho + \left(\frac{\partial p}{\partial s}\right)_\rho \nabla\delta s\right\} = -\frac{c^2}{\rho_0}\nabla\delta\rho - \rho_0\left(\frac{\partial T}{\partial \rho}\right)_s \nabla\delta s$$

Digression -- from the first law of thermodynamics:

$$d\epsilon = Tds + \frac{p}{\rho^2}d\rho \qquad \left(\frac{\partial}{\partial\rho}\left(\frac{\partial\epsilon}{\partial s}\right)_\rho\right)_s = \left(\frac{\partial T}{\partial\rho}\right)_s \quad \Leftrightarrow \quad \left(\frac{\partial}{\partial s}\left(\frac{\partial\epsilon}{\partial\rho}\right)_s\right)_\rho = \left(\frac{\partial p/\rho^2}{\partial s}\right)_\rho \approx \frac{1}{\rho_0^2}\left(\frac{\partial p}{\partial s}\right)_\rho$$

$$\Rightarrow \frac{1}{\rho_0}\left(\frac{\partial p}{\partial s}\right)_\rho = \rho_0\left(\frac{\partial T}{\partial \rho}\right)_s$$

Linearized equations -- continued

$$\frac{\partial \rho}{\partial t} + \nabla\cdot\left(\rho\mathbf{v}\right) = 0$$

$$\Rightarrow \frac{\partial \delta\rho}{\partial t} + \rho_0\nabla\cdot\left(\delta\mathbf{v}\right) = 0$$

$$\rho T\frac{\partial s}{\partial t} = k_{th}\nabla^2 T$$

$$\Rightarrow \frac{\partial \delta s}{\partial t} = \frac{k_{th}}{\rho_0 T_0}\left(\left(\frac{\partial T}{\partial s}\right)_\rho \nabla^2\delta s + \left(\frac{\partial T}{\partial \rho}\right)_s \nabla^2\delta\rho\right)$$

Further relationships:

$$\left(\frac{\partial T}{\partial s}\right)_\rho \approx \frac{T_0}{c_v} \qquad \kappa = \frac{k_{th}}{\rho c_p}$$

heat capacity at constant volume

Linearized equations -- continued

$$\rho T\frac{\partial s}{\partial t} = k_{th}\nabla^2 T$$

$$\Rightarrow \frac{\partial \delta s}{\partial t} = \frac{k_{th}}{\rho_0 T_0}\left(\left(\frac{\partial T}{\partial s}\right)_\rho \nabla^2\delta s + \left(\frac{\partial T}{\partial \rho}\right)_s \nabla^2\delta\rho\right)$$

Further relationships:

$$\left(\frac{\partial T}{\partial s}\right)_\rho \approx \frac{T_0}{c_v} \qquad \kappa = \frac{k_{th}}{\rho c_p}$$

$$\Rightarrow \frac{\partial \delta s}{\partial t} = \left(\gamma\kappa\nabla^2\delta s + \frac{c_p\kappa}{T_0}\left(\frac{\partial T}{\partial \rho}\right)_s \nabla^2\delta\rho\right) \quad \text{where } \gamma \equiv \frac{c_p}{c_v}$$

Newton-Euler equations for viscous fluids – effects on sound

Linearized equations (with the help of various thermodynamic relationships):

$$\frac{\partial \delta\mathbf{v}}{\partial t} = -\frac{c^2}{\rho_0}\nabla\delta\rho - \rho_0\left(\frac{\partial T}{\partial \rho}\right)_s \nabla\delta s + \frac{\eta}{\rho_0}\nabla^2\delta\mathbf{v} + \frac{1}{\rho_0}\left(\zeta + \frac{1}{3}\eta\right)\nabla\left(\nabla\cdot\delta\mathbf{v}\right)$$

$$\frac{\partial \delta\rho}{\partial t} + \rho_0\nabla\cdot\left(\delta\mathbf{v}\right) = 0$$

$$\frac{\partial \delta s}{\partial t} = \gamma\kappa\nabla^2\delta s + \frac{c_p\kappa}{T_0}\left(\frac{\partial T}{\partial \rho}\right)_s \nabla^2\delta\rho$$

Here: $\gamma = \frac{c_p}{c_v}$ $\qquad \kappa = \frac{k_{th}}{c_p\rho_0}$

Linearized hydrodynamic equations

$$\frac{\partial \delta \mathbf{v}}{\partial t} = -\frac{c^2}{\rho_0}\nabla \delta\rho - \rho_0\left(\frac{\partial T}{\partial \rho}\right)_s \nabla \delta s + \frac{\eta}{\rho_0}\nabla^2 \delta\mathbf{v} + \frac{1}{\rho_0}\left(\zeta + \frac{1}{3}\eta\right)\nabla\left(\nabla\cdot\delta\mathbf{v}\right)$$

$$\frac{\partial \delta\rho}{\partial t} + \rho_0 \nabla\cdot\left(\delta\mathbf{v}\right) = 0$$

$$\frac{\partial \delta s}{\partial t} = \gamma\kappa\nabla^2\delta s + \frac{c_p\kappa}{T_0}\left(\frac{\partial T}{\partial \rho}\right)_s \nabla^2\delta\rho$$

It can be shown that

$$\left(\frac{\partial T}{\partial \rho}\right)_s = \frac{Tc^2\beta}{\rho c_p} \approx \frac{T_0 c^2 \beta}{\rho_0 c_p} \qquad \text{where} \quad \beta \equiv \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p \quad \text{(thermal expansion)}$$

Let $\delta\mathbf{v} \equiv \delta\mathbf{v}_0\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$ $\quad \delta\rho \equiv \delta\rho_0\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$ $\quad \delta s \equiv \delta s_0\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$

Linearized hydrodynamic equations; plane wave solutions:

$$\omega\delta\mathbf{v}_0 = \frac{c^2\delta\rho_0}{\rho_0}\mathbf{k} + \frac{T_0\beta c^2}{c_p}\delta s_0\mathbf{k} - \frac{i\eta k^2}{\rho_0}\delta\mathbf{v}_0 - \frac{i}{\rho_0}\left(\zeta+\frac{1}{3}\eta\right)\mathbf{k}\left(\mathbf{k}\cdot\delta\mathbf{v}_0\right)$$

$$\omega\delta\rho_0 - \rho_0\ \mathbf{k}\cdot\delta\mathbf{v}_0 = 0$$

$$\omega\delta s_0 = -i\gamma\kappa k^2\delta s_0 - \frac{i\kappa\beta c^2}{\rho_0}k^2\delta\rho_0$$

In the absense of thermal expansion, $\beta = 0$

$$\omega\delta\mathbf{v}_0 = \frac{c^2\delta\rho_0}{\rho_0}\mathbf{k} - \frac{i\eta k^2}{\rho_0}\delta\mathbf{v}_0 - \frac{i}{\rho_0}\left(\zeta+\frac{1}{3}\eta\right)\mathbf{k}\left(\mathbf{k}\cdot\delta\mathbf{v}_0\right)$$

$$\omega\delta\rho_0 - \rho_0\ \mathbf{k}\cdot\delta\mathbf{v}_0 = 0$$

$$\omega\delta s_0 = -i\gamma\kappa k^2\delta s_0$$

➔Entropy and mechanical modes are independent

In the absense of thermal expansion, $\beta = 0$ -- continued:

$\delta\mathbf{v} \equiv \delta\mathbf{v}_0\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$ $\quad \delta\rho \equiv \delta\rho_0\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$ $\quad \delta s \equiv \delta s_0\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$

$$\omega\delta\mathbf{v}_0 = \frac{c^2\delta\rho_0}{\rho_0}\mathbf{k} - \frac{i\eta k^2}{\rho_0}\delta\mathbf{v}_0 - \frac{i}{\rho_0}\left(\zeta+\frac{1}{3}\eta\right)\mathbf{k}\left(\mathbf{k}\cdot\delta\mathbf{v}_0\right)$$

$$\omega\delta\rho_0 - \rho_0\ \mathbf{k}\cdot\delta\mathbf{v}_0 = 0$$

$$\omega\delta s_0 = -i\gamma\kappa k^2\delta s_0$$

Entropy wave: $\delta s = \delta s_0 e^{-\hat{\mathbf{k}}\cdot\mathbf{r}/\lambda} e^{i(\hat{\mathbf{k}}\cdot\mathbf{r}/\lambda - \omega t)}$ where $\lambda \equiv \sqrt{\frac{2\gamma\kappa}{\omega}}$

Density wave: $\delta\rho_0 = \mathbf{k}\cdot\delta\mathbf{v}_0\,\frac{\rho_0}{\omega}$ $\quad \delta\rho = \delta\rho_0 e^{-\alpha\hat{\mathbf{k}}\cdot\mathbf{r}} e^{i(\omega/c)(\hat{\mathbf{k}}\cdot\mathbf{r}-ct)}$

$$k = \frac{\omega}{c}\left(1 - i\frac{\omega}{c^2\rho_0}\left(\zeta+\frac{3}{4}\eta\right)\right)^{-1/2} \equiv \frac{\omega}{c} + i\alpha \approx \frac{\omega}{c} + i\frac{\omega^2}{2c^3\rho_0}\left(\zeta+\frac{3}{4}\eta\right)$$

Transverse wave: $\delta\mathbf{v}_{0\perp}\times\mathbf{k} \neq 0$ $\quad \delta\mathbf{v}_\perp = \delta\mathbf{v}_{0\perp} e^{-\hat{\mathbf{k}}\cdot\mathbf{r}/\delta} e^{i(\hat{\mathbf{k}}\cdot\mathbf{r}/\delta - \omega t)}$

where $\delta \equiv \sqrt{\frac{2\eta}{\omega\rho_0}}$

Linearized hydrodynamic equations; full plane wave solutions:

$$\omega\delta\mathbf{v}_0 = \frac{c^2\delta\rho_0}{\rho_0}\mathbf{k} + \frac{T_0\beta c^2}{c_p}\delta s_0\mathbf{k} - \frac{i\eta k^2}{\rho_0}\delta\mathbf{v}_0 - \frac{i}{\rho_0}\left(\zeta + \frac{1}{3}\eta\right)\mathbf{k}\left(\mathbf{k}\cdot\delta\mathbf{v}_0\right)$$

$$\omega\delta\rho_0 - \rho_0\ \mathbf{k}\cdot\delta\mathbf{v}_0 = 0$$

$$\omega\delta s_0 = -i\gamma\kappa k^2\delta s_0 - \frac{i\kappa\beta c^2}{\rho_0}k^2\delta\rho_0$$

Transverse modes $(\delta\mathbf{v}\cdot\mathbf{k} = 0)$:

$$\delta\rho_0 = 0 \qquad \delta s_0 = 0$$

$$\left(\omega + \frac{i\eta k^2}{\rho_0}\right)\left(\delta\mathbf{v}\times\ \mathbf{k}\right) = 0 \qquad k = \pm\left(\frac{i\omega\rho_0}{\eta}\right)^{1/2}$$

$$\delta\mathbf{v}_\perp = \delta\mathbf{v}_{0\perp}e^{-\hat{\mathbf{k}}\cdot\mathbf{r}/\delta}e^{i(\hat{\mathbf{k}}\cdot\mathbf{r}/\delta - \omega t)} \qquad \text{where} \quad \delta \equiv \sqrt{\frac{2\eta}{\omega\rho_0}}$$

11/18/2016 PHY 711 Fall 2016 -- Lecture 33 16

Linearized hydrodynamic equations; full plane wave solutions:

$$\omega\delta\mathbf{v}_0 = \frac{c^2\delta\rho_0}{\rho_0}\mathbf{k} + \frac{T_0\beta c^2}{c_p}\delta s_0\mathbf{k} - \frac{i\eta k^2}{\rho_0}\delta\mathbf{v}_0 - \frac{i}{\rho_0}\left(\zeta + \frac{1}{3}\eta\right)\mathbf{k}\left(\mathbf{k}\cdot\delta\mathbf{v}_0\right)$$

$$\omega\delta\rho_0 - \rho_0\ \mathbf{k}\cdot\delta\mathbf{v}_0 = 0$$

$$\omega\delta s_0 = -i\gamma\kappa k^2\delta s_0 - \frac{i\kappa\beta c^2}{\rho_0}k^2\delta\rho_0$$

Longitudinal solutions: $(\delta\mathbf{v}\cdot\mathbf{k} \neq 0)$:

$$\left(\omega^2 - c^2k^2 + i\frac{\omega k^2}{\rho_0}\left(\frac{4}{3}\eta + \zeta\right)\right)\delta\rho_0 - \frac{\rho_0 T_0\beta c^2 k^2}{c_p}\delta s_0 = 0$$

$$\frac{i\kappa\beta c^2}{\rho_0}k^2\delta\rho_0 + \left(\omega + i\gamma\kappa k^2\right)\delta s_0 = 0$$

11/18/2016 PHY 711 Fall 2016 -- Lecture 33 17

Linearized hydrodynamic equations; full plane wave solutions:

Longitudinal solutions: $(\delta\mathbf{v}\cdot\mathbf{k} \neq 0)$:

$$\left(\omega^2 - c^2k^2 + i\frac{\omega k^2}{\rho_0}\left(\frac{4}{3}\eta + \zeta\right)\right)\delta\rho_0 - \frac{\rho_0 T_0\beta c^2 k^2}{c_p}\delta s_0 = 0$$

$$\frac{i\kappa\beta c^2}{\rho_0}k^2\delta\rho_0 + \left(\omega + i\gamma\kappa k^2\right)\delta s_0 = 0$$

Approximate solution: $k = \dfrac{\omega}{c} + i\alpha$

where $\alpha \approx \dfrac{\omega^2}{2c^3\rho_0}\left(\dfrac{4}{3}\eta + \zeta\right) + \dfrac{\kappa T_0\beta^2\omega^2}{2c_p c}$

$$\delta\rho = \delta\rho_0 e^{-\alpha\hat{\mathbf{k}}\cdot\mathbf{r}}e^{i(\omega/c)\left(\hat{\mathbf{k}}\cdot\mathbf{r} - ct\right)}$$

11/18/2016 PHY 711 Fall 2016 -- Lecture 33 18