PHY 711 Classical Mechanics and Mathematical Methods
11-11:50 AM MWF Olin 107

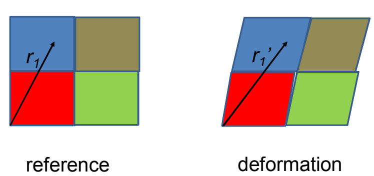
Plan for Lecture 34

Physics of elastic continua – Chap. 13 in F & W

1. **Stress and strain**
2. **Waves in elastic media**

23	Wed, 10/26/2016	Chap. 9	Introduction to hydrodynamics		
24	Fri, 10/28/2016	Chap. 9	Introduction to hydrodynamics	#18	10/31/2016
25	Mon, 10/31/2016	Chap. 9	Sound waves	#19	11/02/2016
26	Wed, 11/02/2016	Chap. 9	Sound waves	#20	11/04/2016
27	Fri, 11/04/2016	Chap. 9	Non-linear sound	#21	11/07/2016
28	Mon, 11/07/2016	Chap. 10	Surface waves in fluids		
29	Wed, 11/09/2016	Chap. 10	Surface waves in fluids	#22	11/11/2016
30	Fri, 11/11/2016	Chap. 11	Heat conductivity	#23	11/14/2016
31	Mon, 11/14/2016	Chap. 12	Viscous fluids	#24	11/16/2016
32	Wed, 11/16/2016	Chap. 12	Viscous fluids	#25	11/18/2016
33	Fri, 11/18/2016	Chap. 12	Viscous fluids	#26	11/21/2016
34	Mon, 11/21/2016	Chap. 13	Elastic continua	Prepare presentations	
	Wed, 11/23/2016		Thanksgiving Holiday – no class		
	Fri, 11/25/2016		Thanksgiving Holiday – no class		
35	Mon, 11/28/2016				
36	Wed, 11/30/2016				
37	Fri, 12/02/2016				
38	Mon, 12/05/2016				
	Wed, 12/07/2016		Presentations I		
	Fri, 12/09/2016		Presentations II		

Brief introduction to elastic continua



$$\mathbf{r}_1' = \mathbf{r}_1 + \mathbf{u}(\mathbf{r}_1) \qquad \mathbf{r}_2' = \mathbf{r}_2 + \mathbf{u}(\mathbf{r}_2)$$

$$\mathbf{r}_2' - \mathbf{r}_1' = \mathbf{r}_2 - \mathbf{r}_1 + \left((\mathbf{r}_2 - \mathbf{r}_1) \cdot \nabla \right) \mathbf{u}(\mathbf{r}_1) + \ldots$$

Brief introduction to elastic continua -- continued

Deformation components:

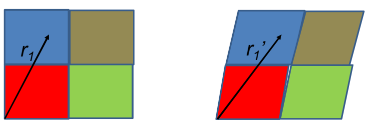
$$\frac{\partial u_i}{\partial x_j} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$

$$\equiv \epsilon_{ij} + O_{ij}$$

elastic strain tensor (ϵ_{ij})

rotation of material (O_{ij})

Brief introduction to elastic continua -- continued



reference deformation

$$\mathbf{r}_1' = \mathbf{r}_1 + \mathbf{u}(\mathbf{r}_1) \qquad \mathbf{r}_2' = \mathbf{r}_2 + \mathbf{u}(\mathbf{r}_2)$$

$$\mathbf{r}_2' - \mathbf{r}_1' = \mathbf{r}_2 - \mathbf{r}_1 + \left((\mathbf{r}_2 - \mathbf{r}_1)\cdot\nabla\right)\mathbf{u}(\mathbf{r}_1) + \ldots$$

$$x_{2i}' - x_{1i}' \approx x_{2i} - x_{1i} + \sum_{j=1}^{3} \epsilon_{ij}\left(x_{2j} - x_{1j}\right)$$

Effects of strain on a vector:

$$\mathbf{a}' = \mathbf{a} + a\left(\epsilon_{11}\hat{\mathbf{x}} + \epsilon_{21}\hat{\mathbf{y}} + \epsilon_{31}\hat{\mathbf{z}}\right)$$

$$a' = |\mathbf{a}'\cdot\mathbf{a}'| \approx a\left(1 + \epsilon_{11}\right)$$

Deformation

$$\mathbf{a}' = \mathbf{a} + a\left(\epsilon_{11}\hat{\mathbf{x}} + \epsilon_{21}\hat{\mathbf{y}} + \epsilon_{31}\hat{\mathbf{z}}\right)$$

$$\mathbf{b}' = \mathbf{b} + b\left(\epsilon_{12}\hat{\mathbf{x}} + \epsilon_{22}\hat{\mathbf{y}} + \epsilon_{32}\hat{\mathbf{z}}\right)$$

$$\text{for } \mathbf{a}\cdot\mathbf{b} = 0 = ab\cos\theta \qquad \Rightarrow \theta = \frac{\pi}{2}$$

$$\mathbf{a}'\cdot\mathbf{b}' \approx ab\left(\epsilon_{21} + \epsilon_{12}\right) = 2ab\epsilon_{12} = ab\cos\theta'$$

$$\cos\theta' = \cos\left(\theta + (\theta' - \theta)\right) = \cos\theta\,\cos(\theta' - \theta) - \sin\theta\,\sin(\theta' - \theta)$$

$$\approx -\sin\theta\,\sin(\theta' - \theta) \approx -(\theta' - \theta)$$

$$\theta' \approx \theta - 2\epsilon_{12} = \frac{\pi}{2} - 2\epsilon_{12}$$

Brief introduction to elastic continua -- continued

Deformation components:

$$\frac{\partial u_i}{\partial x_j} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$

$$\equiv \epsilon_{ij} + \Omega_{ij}$$

$V = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ $\quad V' = \mathbf{a}'\cdot(\mathbf{b}'\times\mathbf{c}')$ $\quad V' = V(1+\nabla\cdot\mathbf{u}) = V\left(1+\mathrm{Tr}(\epsilon)\right)$

$$\nabla\cdot\mathbf{u} = \epsilon_{11} + \epsilon_{22} + \epsilon_{33} = \mathrm{Tr}(\epsilon) == \frac{dV}{V} = -\frac{d\rho}{\rho}$$

Elastic stress tensor

$-\sum_{j=1}^{3} T_{ij}\, dA_j \Rightarrow i^{th}$ component of force acting on surface $\hat{\mathbf{n}}dA \equiv d\mathbf{A}$

Generalization of Hooke's law, $\mathrm{F}_x = -kx$:

Lame' coefficients : $T_{ij} = -\lambda\delta_{ij}\nabla\cdot\mathbf{u} - \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$

or : $T_{ij} = -\lambda\delta_{ij}\mathrm{Tr}(\epsilon) - 2\mu\epsilon_{ij}$

Elastic stress tensor -- continued

$$T_{ij} = -\lambda\delta_{ij}\mathrm{Tr}(\epsilon) - 2\mu\epsilon_{ij}$$

Note that: $\mathrm{Tr}(T) = -3\underbrace{\left(\lambda + \frac{2}{3}\mu\right)}_{K}\mathrm{Tr}(\epsilon)$

$K \equiv$ bulk modulus$= -V\left(\frac{\partial p}{\partial V}\right)$

Inverse Hooke's law: $\epsilon_{ij} = -\frac{1}{2\mu}\left(T_{ij} - \frac{\lambda}{3\left(\lambda + \frac{2}{3}\mu\right)}\delta_{ij}Tr(T)\right)$

Stress tensor -- continued

In terms of bulk modulus: $K = \lambda + \frac{2}{3}\mu$

$$\epsilon_{ij} = -\frac{1}{2\mu}\left(T_{ij} - \frac{\lambda}{3\left(\lambda + \frac{2}{3}\mu\right)}\,\delta_{ij}Tr(T)\right)$$

$$\epsilon_{ij} = -\frac{1}{9K}\delta_{ij}Tr(T) - \frac{1}{2\mu}\left(T_{ij} - \frac{1}{3}\,\delta_{ij}Tr(T)\right)$$

$$\epsilon_{ij} = -\frac{1}{2\mu}\left(T_{ij} - \frac{\lambda}{3\left(\lambda + \frac{2}{3}\mu\right)}\,\delta_{ij}Tr(T)\right) = -\frac{1}{9K}\delta_{ij}Tr(T) - \frac{1}{2\mu}\left(T_{ij} - \frac{1}{3}\,\delta_{ij}Tr(T)\right)$$

Example -- hydrostatic pressure: $T_{ij} = \delta_{ij}dp$

$$\mathrm{Tr}(T) = 3dp$$

$$\epsilon_{ij} = -\frac{dp}{3\left(\lambda + \frac{2}{3}\mu\right)}\delta_{ij} \equiv -\frac{dp}{3K}\delta_{ij}$$

Note that: $\mathrm{Tr}(\epsilon) = \frac{dV}{V} = -\frac{dp}{K}$

$$\Rightarrow K = -V\frac{\partial p}{\partial V}$$

$$\epsilon_{ij} = -\frac{1}{2\mu}\left(T_{ij} - \frac{\lambda}{3\left(\lambda + \frac{2}{3}\mu\right)}\,\delta_{ij}Tr(T)\right) = -\frac{1}{9K}\delta_{ij}Tr(T) - \frac{1}{2\mu}\left(T_{ij} - \frac{1}{3}\,\delta_{ij}Tr(T)\right)$$

Example -- uniaxial pressure: $T_{ij} = \begin{cases} dp & ij = zz \\ 0 & \text{otherwise} \end{cases}$

$\epsilon_{zz} = -\frac{1}{E}T_{zz}$ in terms of Young's modulus

$$E = \frac{9K\mu}{3K + \mu}$$

$$\epsilon_{xx} = \epsilon_{yy} = -\left(\frac{1}{9K} - \frac{1}{6\mu}\right)dp$$

Poisson ratio: $\sigma = -\frac{\epsilon_{xx}}{\epsilon_{zz}} = -\frac{\epsilon_{yy}}{\epsilon_{zz}} = \frac{1}{2}\frac{3K - 2\mu}{3K + \mu}$

Example -- uniaxial pressure: $T_{ij} = \begin{cases} dp & ij = zz \\ 0 & \text{otherwise} \end{cases}$

transverse contributions:

$$\epsilon_{xx} = \epsilon_{yy} = \left(-\frac{1}{9K} + \frac{1}{6\mu} \right) T_{zz}$$

Poisson's ratio:

$$\sigma = -\frac{\epsilon_{xx}}{\epsilon_{zz}} = \frac{1}{2}\frac{3K - 2\mu}{3K + \mu}$$

Relationships between elastic constants:

$$K = \frac{1}{3}\frac{E}{1-2\sigma}$$

$$\mu = \frac{1}{2}\frac{E}{1+\sigma}$$

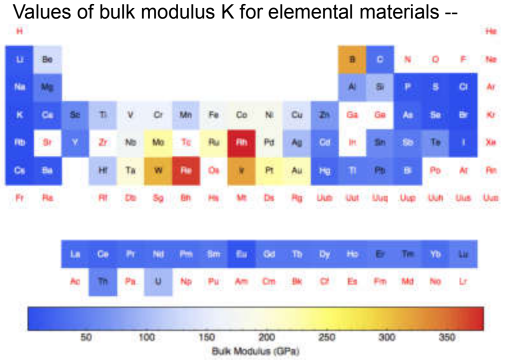
$$\epsilon_{ij} = -\frac{1}{9K}\delta_{ij}Tr(T) - \frac{1}{2\mu}\left(T_{ij} - \frac{1}{3}\,\delta_{ij}Tr(T) \right)$$

Shear modulus

$$T_{ij} = \begin{cases} -f & \text{for } T_{xy} \text{ or } T_{yx} \\ 0 & \text{otherwise} \end{cases}$$

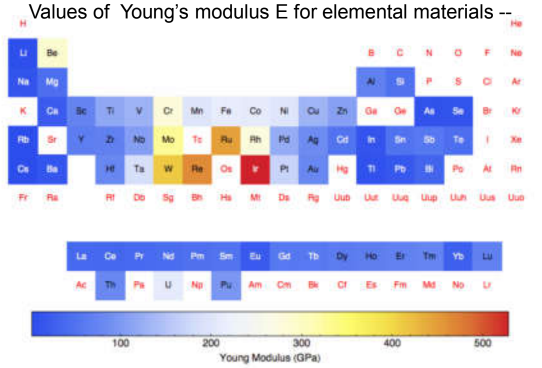
$$\epsilon_{xy} = \epsilon_{yx} = \frac{f}{\mu}$$

Values of bulk modulus K for elemental materials --



Up to date, curated data provided by *Mathematica*'s ElementData function from Wolfram Research, Inc.

Values of Young's modulus E for elemental materials --



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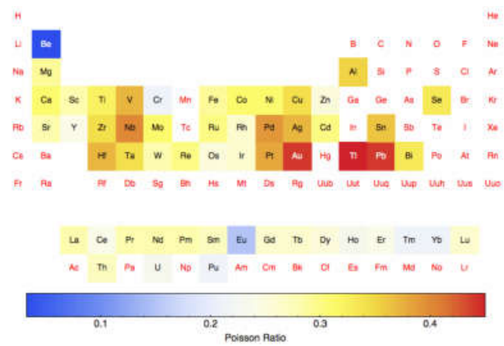
Values of Poisson ratio σ for elemental materials --



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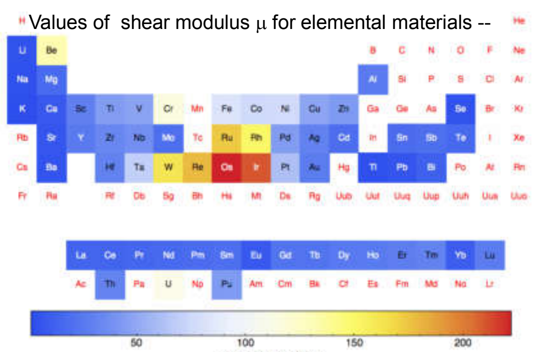
Values of shear modulus μ for elemental materials --



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Dynamical equations of elastic continuum

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = \mu \nabla^2 \mathbf{u} + \left(K + \frac{1}{3}\mu \right) \nabla \left(\nabla \cdot \mathbf{u} \right) + \rho \mathbf{f}$$

In the absence of external forces, this reduces to two decoupled wave equations representing longitudinal and transverse modes:

$\mathbf{u} = \mathbf{u}_l + \mathbf{u}_t$

where $\nabla \times \mathbf{u}_l = 0$ and $\nabla \cdot \mathbf{u}_t = 0$

$$c_l = \left(\frac{K + \frac{4}{3}\mu}{\rho} \right)^{1/2} \quad \text{and} \quad c_t = \left(\frac{\mu}{\rho} \right)^{1/2}$$

Typical velocities of longitudinal sound waves

http://www.engineeringtoolbox.com/sound-speed-solids-d_713.html

air: c_l=343 m/s
water: c_l=1433 m/s

Material	c_l (m/s)
Aluminum, shear - longitudinal wave	3100 - 6400
Beryllium	12890
Brass	3475
Brick	4176
Concrete	3200 - 3600
Copper	4600
Cork	366 - 518
Diamond	12000
Glass	3962
Glass, Pyrex	5640
Gold	3240
Granite	5950
Hardwood	3962
Iron	5130
Lead	1960 - 2160
Lucite	2680
Rubber, butyl	1830
Rubber	40 - 150
Silver	3650
Steel	6100
Steel, stainless	5790
Titanium	6070
Wood (hard)	3960
Wood	3300 - 3600

from:
https://pangea.stanford.edu/courses/gp262/Notes/5.Elasticity.pdf

Mineral	Density	Young's Modulus	Bulk Modulus	Shear Modulus	Vp	Vs	Poisson's Ratio
Quartz	2.6500	95.756	36.600	45.000	6.0376	4.1208	0.063953
Calcite	2.7100	84.293	76.800	32.000	6.6395	3.4363	0.31707
Dolomite	2.8700	116.57	94.900	45.000	7.3465	3.9597	0.29527
Clay (kaolinite)	1.5800	3.2034	1.5000	1.4000	1.4597	0.94132	0.14407
Muscovite	2.7900	100.84	61.500	41.100	6.4563	3.8381	0.22673
Feldspar (Albite)	2.6300	69.010	75.600	25.600	6.4594	3.1199	0.34786
Halite	2.1600	37.242	24.800	14.900	4.5474	2.6264	0.24972
Anhydrite	2.9800	74.431	56.100	29.100	5.6432	3.1249	0.27888
Pyrite	4.9300	305.85	147.40	132.50	8.1076	5.1842	0.15417
Siderite	3.9600	134.51	123.70	51.000	6.9576	3.5887	0.31876
gas	0.00065000	0.0000	0.00013000	0.0000	0.44721	0.0000	0.50000
water	1.0000	0.0000	2.2500	0.0000	1.5000	0.0000	0.50000
oil	0.80000	0.0000	1.0200	0.0000	1.1292	0.0000	0.50000

C.1

densities in g/cm^3
moduli in GPa
velocities in km/s