PHY 711 Classical Mechanics and Mathematical Methods
11-11:50 AM MWF Olin 107

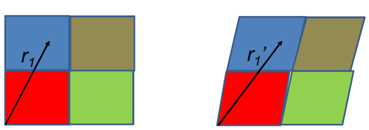
Plan for Lecture 35

Physics of elastic continua – Chap. 13 in F & W

1. **Elastic energy**
2. **Waves in elastic media**

23	Wed, 10/26/2016	Chap. 9	Introduction to hydrodynamics		
24	Fri, 10/28/2016	Chap. 9	Introduction to hydrodynamics	#18	10/31/2016
25	Mon, 10/31/2016	Chap. 9	Sound waves	#19	11/02/2016
26	Wed, 11/02/2016	Chap. 9	Sound waves	#20	11/04/2016
27	Fri, 11/04/2016	Chap. 9	Non-linear sound	#21	11/07/2016
28	Mon, 11/07/2016	Chap. 10	Surface waves in fluids		
29	Wed, 11/09/2016	Chap. 10	Surface waves in fluids	#22	11/11/2016
30	Fri, 11/11/2016	Chap. 11	Heat conductivity	#23	11/14/2016
31	Mon, 11/14/2016	Chap. 12	Viscous fluids	#24	11/16/2016
32	Wed, 11/16/2016	Chap. 12	Viscous fluids	#25	11/18/2016
33	Fri, 11/18/2016	Chap. 12	Viscous fluids	#26	11/21/2016
34	Mon, 11/21/2016	Chap. 13	Elastic continua	Prepare presentations	
	Wed, 11/23/2016		Thanksgiving Holiday -- no class		
	Fri, 11/25/2016		Thanksgiving Holiday -- no class		
35	Mon, 11/28/2016	Chap. 13	Elastic continua	Prepare presentations	
36	Wed, 11/30/2016		Math methods	Prepare presentations	
37	Fri, 12/02/2016		Math methods	Prepare presentations	
38	Mon, 12/05/2016		Math methods	Prepare presentations	
	Wed, 12/07/2016		Presentations I		
	Fri, 12/09/2016		Presentations II		

Brief introduction to elastic continua



reference deformation

$$\mathbf{r}_1' = \mathbf{r}_1 + \mathbf{u}(\mathbf{r}_1) \qquad \mathbf{r}_2' = \mathbf{r}_2 + \mathbf{u}(\mathbf{r}_2)$$

$$\mathbf{r}_2' - \mathbf{r}_1' = \mathbf{r}_2 - \mathbf{r}_1 + \left((\mathbf{r}_2 - \mathbf{r}_1)\cdot\nabla\right)\mathbf{u}(\mathbf{r}_1) + \ldots$$

For cartesian components $i = 1,2,3$:

$$x_{2i}' - x_{1i}' \approx x_{2i} - x_{1i} + \sum_{j=1}^{3} \epsilon_{ij}\left(x_{2j} - x_{1j}\right)$$

Where

$$\epsilon_{ij} \equiv \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

Elastic stress tensor

$$-\sum_{j=1}^{3} T_{ij} dA_j \Rightarrow i^{th} \text{ component of force acting on surface } \hat{\mathbf{n}}dA \equiv d\mathbf{A}$$

Generalization of Hooke's law, $F_x = -kx$:

$$\text{Lame' coefficients : } \quad T_{ij} = -\lambda\delta_{ij}\nabla\cdot\mathbf{u} - \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = -\lambda\delta_{ij}\mathrm{Tr}(\epsilon) - 2\mu\epsilon_{ij}$$

material-dependent empirical parameters

Bulk modulus: $K = \lambda + \frac{2}{3}\mu$

Young's modulus: $E = \frac{9K\mu}{3K+\mu}$

Poisson ratio: $\sigma = \frac{1}{2}\frac{3K-2\mu}{3K+\mu}$

Shear modulus: μ

Stress

Strain

$$\text{Hooke's Law:} \quad T_{ij} = -K\delta_{ij}\mathrm{Tr}(\epsilon) - 2\mu\left(\epsilon_{ij} - \frac{1}{3}\delta_{ij}\mathrm{Tr}(\epsilon)\right)$$

$$\text{Inverse Hooke's Law:} \quad \epsilon_{ij} = -\frac{1}{9K}\delta_{ij}Tr(T) - \frac{1}{2\mu}\left(T_{ij} - \frac{1}{3}\delta_{ij}Tr(T)\right)$$

$$\text{Elastic force:} \quad F_i^{\text{elastic}} = \sum_{j=1}^{3}\frac{\partial T_{ij}}{\partial x_j}$$

Elastic work due to distortion $\delta\mathbf{u}$

$$\delta W^{\text{elastic}} = \int_V d^3x\ \mathbf{F}^{\text{elastic}}\cdot\delta\mathbf{u}$$

$$= \int_V d^3x \sum_{ij=1}^{3}\frac{\partial T_{ij}}{\partial x_j}\delta u_i$$

$$= \sum_{ij=1}^{3}\int_A dA_j\ T_{ij}\delta u_i - \sum_{ij=1}^{3}\int_V d^3x T_{ij}\frac{\partial \delta u_i}{\partial x_j}$$

Elastic work due to distortion $\delta\mathbf{u}$

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$$= \sum_{ij=1}^{3}\int_A dA_j\ T_{ij}\delta u_i - \sum_{ij=1}^{3}\int_V d^3x T_{ij}\epsilon_{ij}$$

surface contribution

bulk contribution

$$\text{For large samples:} \quad \delta W^{\text{elastic}} = -\sum_{ij=1}^{3}\int_V d^3x T_{ij}\epsilon_{ij}$$

Bulk elastic energy: $\delta W^{\text{elastic}} = -\sum_{ij=1}^{3} \int_V d^3x T_{ij} \epsilon_{ij}$

Integrating from 0 to final strain ϵ_{ij} :

$$\delta W^{\text{elastic}} = -\frac{1}{2} \sum_{ij=1}^{3} T_{ij} \epsilon_{ij}$$

Hooke's Law: $T_{ij} = -K\delta_{ij}\text{Tr}(\epsilon) - 2\mu\left(\epsilon_{ij} - \frac{1}{3}\delta_{ij}\text{Tr}(\epsilon)\right)$

$$\delta W^{\text{elastic}} = \frac{1}{2} K (\text{Tr }\epsilon)^2 + \mu \sum_{ij=1}^{3} \left(\epsilon_{ij} - \frac{1}{3}\text{Tr }\epsilon\right)^2$$
$$= \frac{1}{2}\lambda(\text{Tr }\epsilon)^2 + \mu \sum_{ij=1}^{3} \epsilon_{ij}^2$$

Note that the two relations:

$$\delta W^{\text{elastic}} = \frac{1}{2} K (\text{Tr }\epsilon)^2 + \mu \sum_{ij=1}^{3} \left(\epsilon_{ij} - \frac{1}{3}\text{Tr }\epsilon\right)^2$$

$$T_{ij} = -K\delta_{ij}\text{Tr}(\epsilon) - 2\mu\left(\epsilon_{ij} - \frac{1}{3}\delta_{ij}\text{Tr}(\epsilon)\right)$$

Ensure that: $\frac{\partial \delta W^{\text{elastic}}}{\partial \epsilon_{ij}} = -T_{ij}$

Dynamical equations of motion

Recall Newton's second law for continuum system with density ρ, velocity components v_k and stress tensor T_{ij} :

$$\frac{\partial(\rho v_k)}{\partial t} = -\sum_{l=1}^{3} \frac{\partial T_{kl}}{\partial x_l} + \rho f_k$$

external force density

For our elastic medium: ρ does not vary in time

Velocity related to displacement: $v_k = \frac{\partial u_k}{\partial t}$

Hooke's law: $T_{kl} = -K\delta_{kl}\text{Tr}(\epsilon) - 2\mu\left(\epsilon_{kl} - \frac{1}{3}\delta_{kl}\text{Tr}(\epsilon)\right)$

$$= -K\delta_{kl}(\nabla\cdot\mathbf{u}) - 2\mu\left(\frac{1}{2}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right) - \frac{1}{3}\delta_{kl}(\nabla\cdot\mathbf{u})\right)$$

For: $T_{kl} = -K\delta_{kl}(\nabla\cdot\mathbf{u}) - 2\mu\left(\frac{1}{2}\left(\frac{\partial u_k}{\partial x_l}+\frac{\partial u_l}{\partial x_k}\right)-\frac{1}{3}\delta_{kl}(\nabla\cdot\mathbf{u})\right)$

$$\sum_{l=1}^{3}\frac{\partial T_{kl}}{\partial x_l} = -\left(K-\frac{2}{3}\mu\right)\sum_{l=1}^{3}\left(\delta_{kl}\frac{\partial(\nabla\cdot\mathbf{u})}{\partial x_l}\right) - \mu\sum_{l=1}^{3}\frac{\partial^2 u_l}{\partial x_k\partial x_l} - \mu\sum_{l=1}^{3}\frac{\partial^2 u_k}{\partial^2 x_l}$$

$$= -\left(K+\frac{1}{3}\mu\right)\sum_{l=1}^{3}\frac{\partial^2 u_l}{\partial x_k\partial x_l} - \mu\sum_{l=1}^{3}\frac{\partial^2 u_k}{\partial^2 x_l}$$

$$\frac{\partial(\rho v_k)}{\partial t} = -\sum_{l=1}^{3}\frac{\partial T_{kl}}{\partial x_l} + \rho f_k$$

$$\rho\frac{\partial^2 u_k}{\partial t^2} = \left(K+\frac{1}{3}\mu\right)\sum_{l=1}^{3}\frac{\partial^2 u_l}{\partial x_k\partial x_l} + \mu\sum_{l=1}^{3}\frac{\partial^2 u_k}{\partial^2 x_l} + \rho f_k$$

Vector form: $\rho\frac{\partial^2\mathbf{u}}{\partial t^2} = \left(K+\frac{1}{3}\mu\right)\nabla(\nabla\cdot\mathbf{u}) + \mu\nabla^2\mathbf{u} + \rho\mathbf{f}$

Dynamical equations of elastic continuum

$$\rho\frac{\partial^2\mathbf{u}}{\partial t^2} = \mu\nabla^2\mathbf{u} + \left(K+\frac{1}{3}\mu\right)\nabla(\nabla\cdot\mathbf{u}) + \rho\mathbf{f}$$

In absense of external force:

$$\rho\frac{\partial^2\mathbf{u}}{\partial t^2} = \mu\nabla^2\mathbf{u} + \left(K+\frac{1}{3}\mu\right)\nabla(\nabla\cdot\mathbf{u})$$

Suppose: $\mathbf{u}=\mathbf{u}_l+\mathbf{u}_t$

where $\nabla\times\mathbf{u}_l = 0$ and $\nabla\cdot\mathbf{u}_t = 0$

$$\rho\frac{\partial^2(\mathbf{u}_l+\mathbf{u}_t)}{\partial t^2} = \mu\nabla^2(\mathbf{u}_l+\mathbf{u}_t) + \left(K+\frac{1}{3}\mu\right)\nabla(\nabla\cdot\mathbf{u}_l)$$

Dynamical equations of elastic continuum

Transverse component:

$$\rho\frac{\partial^2\mathbf{u}_t}{\partial t^2} = \mu\nabla^2\mathbf{u}_t \qquad \frac{\partial^2\mathbf{u}_t}{\partial t^2} = \frac{\mu}{\rho}\nabla^2\mathbf{u}_t \equiv c_t^2\nabla^2\mathbf{u}_t$$

Transverse wave velocity: $c_t = \sqrt{\frac{\mu}{\rho}}$

Longitudinal component: $\rho\frac{\partial^2\mathbf{u}_l}{\partial t^2} = \mu\nabla^2\mathbf{u}_l + \left(K+\frac{1}{3}\mu\right)\nabla(\nabla\cdot\mathbf{u}_l)$

Note that the longitudinal wave has its displacement along its propagation direction x_p, so that $\mathbf{u}_l = \mathbf{u}_l(x_p) \equiv u_l(x_p)\hat{\mathbf{x}}_p$

$$\Rightarrow \rho\frac{\partial^2 u_l}{\partial t^2} = \mu\frac{\partial^2 u_l}{\partial x_p^2} + \left(K+\frac{1}{3}\mu\right)\frac{\partial^2 u_l}{\partial x_p^2} = \left(K+\frac{4}{3}\mu\right)\frac{\partial^2 u_l}{\partial x_p^2}$$

Dynamical equations of elastic continuum
Longitudinal component -- continued:

$$\rho \frac{\partial^2 u_l}{\partial t^2} = \left(K + \frac{4}{3}\mu \right) \frac{\partial^2 u_l}{\partial x_p^2}$$

$$\frac{\partial^2 u_l}{\partial t^2} = \left(\frac{K}{\rho} + \frac{4}{3}\frac{\mu}{\rho} \right) \frac{\partial^2 u_l}{\partial x_p^2} \equiv c_l^2 \frac{\partial^2 u_l}{\partial x_p^2}$$

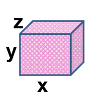
Longitudinal wave velocity: $\mathrm{c}_l = \sqrt{\frac{K}{\rho} + \frac{4}{3}\frac{\mu}{\rho}}$

Some values:

Substance	Density (g/cm^3)	V_l (m/s)	V_S (m/s)
Metals			
Aluminum, rolled	2.7	6420	3040
Berylium	1.87	12890	8880
Brass (70 Cu, 30 Zn)	8.6	4700	2110
Copper, annealed	8.93	4760	2325
Copper, rolled	8.93	5010	2270
Gold, hard-drawn	19.7	3240	1200
Iron, Armco	7.85	5960	3240
Lead, annealed	11.4	2160	700
Lead, rolled	11.4	1960	690
Molybdenum	10.1	6250	3350
Monel metal	8.90	5350	2720
Nickel (unmagnetized)	8.85	5480	2990
Nickel	8.9	6040	3000
Platinum	21.4	3260	1730
Silver	10.4	3650	1610
Steel, mild	7.85	5960	3235
Steel, 347 Stainless	7.9	5790	3100

Substance	Density (g/cm^3)	V_l (m/s)	V_S (m/s)
Tin, rolled	7.3	3320	1670
Titaninum	4.5	6070	3125
Tungsten, annealed	19.3	5220	2890
Tungsten Carbide	13.8	6655	3980
Zinc, rolled	7.1	4210	2440
Various			
Fused silica	2.2	5968	3764
Glass, pyrex	2.32	5640	3280
Glass, heavy silicate flint	3.88	3980	2380
Lucite	1.18	2680	1100
Nylon 6-6	1.11	2620	1070
Polyethylene	0.90	1950	540
Polystyrene	1.06	2350	1120

Elasticity in solids



We imagine that near equilibrium, the solid can be described in terms of a potential function $\phi(\{\mathbf{R}\})$ where $\{\mathbf{R}\}$ represents the positions of each atom:

$$E = \sum_{\mathbf{R}} \phi(\{\mathbf{R}\}) + \frac{1}{2}\sum_{\mathbf{RR}'} (\mathbf{u}(\mathbf{R}) - \mathbf{u}(\mathbf{R}')) \cdot \nabla \phi(\{\mathbf{R}\}) + \frac{1}{4}\sum_{\mathbf{RR}'} \left((\mathbf{u}(\mathbf{R}) - \mathbf{u}(\mathbf{R}')) \cdot \nabla \right)^2 \phi(\{\mathbf{R}\}) + \ldots$$

vanishes at equilibrium

$$\delta W^{\text{elastic}} = \frac{1}{4} \sum_{\substack{\mathbf{RR}' \\ ij}} \left(u_i(\mathbf{R}) - u_i(\mathbf{R}') \right) \frac{\partial^2 \phi(\{\mathbf{R}\})}{\partial u_i \partial u_j} \left(u_j(\mathbf{R}) - u_j(\mathbf{R}') \right)$$

$$\delta W^{\text{elastic}} = \frac{1}{4}\sum_{\substack{\mathbf{RR}'\\ ij}} \left(u_i(\mathbf{R}) - u_i(\mathbf{R}')\right) \frac{\partial^2 \phi(\{\mathbf{R}\})}{\partial u_i \partial u_j} \left(u_j(\mathbf{R}) - u_j(\mathbf{R}')\right)$$

Note that $\mathbf{u}(\mathbf{R}') \approx \mathbf{u}(\mathbf{R}') + (\mathbf{R}' - \mathbf{R}) \cdot \nabla \mathbf{u}(\mathbf{R})$

In terms of strain coeffients ϵ_{ij} :

$$\delta W^{\text{elastic}} = \frac{1}{2}\sum_{ijkl} \epsilon_{ij} c_{ijkl} \epsilon_{kl}$$

where coeffients c_{ijkl} are composed of permutations of $R_i \dfrac{\partial^2 \phi}{\partial u_j \partial u_k} R_l$

For the most general case c_{ijkl} have 21 distrinct terms, for a cube there are only 3 unique terms.

Simplified notation:

$xx \rightarrow 1$
$yy \rightarrow 2$
$zz \rightarrow 3$
$yz \rightarrow 4$
$zx \rightarrow 5$
$xy \rightarrow 6$

For cubic crystals, the unique coefficients are:

$C_{11} = c_{xxxx}$ $\quad C_{12} = c_{xxyy}$ $\quad C_{44} = c_{yzyz}$

Some typical values (Ref. Ashcroft and Mermin (1976))

	C_{11} (GPa)	C_{12}(GPa)	C_{44}(GPa)
Na	7.0	6.1	4.5
Al	107	61	28
Fe	234	136	118
Si	166	64	80
NaCl	48.7	12.4	12.6