PHY 711 Classical Mechanics and Mathematical Methods
11-11:50 AM MWF Olin 107

Plan for Lecture 36

Review of mathematical methods – complex numbers

23	Wed, 10/26/2016	Chap. 9	Introduction to hydrodynamics		
24	Fri, 10/28/2016	Chap. 9	Introduction to hydrodynamics	#18	10/31/2016
25	Mon, 10/31/2016	Chap. 9	Sound waves	#19	11/02/2016
26	Wed, 11/02/2016	Chap. 9	Sound waves	#20	11/04/2016
27	Fri, 11/04/2016	Chap. 9	Non-linear sound	#21	11/07/2016
28	Mon, 11/07/2016	Chap. 10	Surface waves in fluids		
29	Wed, 11/09/2016	Chap. 10	Surface waves in fluids	#22	11/11/2016
30	Fri, 11/11/2016	Chap. 11	Heat conductivity	#23	11/14/2016
31	Mon, 11/14/2016	Chap. 12	Viscous fluids	#24	11/16/2016
32	Wed, 11/16/2016	Chap. 12	Viscous fluids	#25	11/18/2016
33	Fri, 11/18/2016	Chap. 12	Viscous fluids	#26	11/21/2016
34	Mon, 11/21/2016	Chap. 13	Elastic continua	Prepare presentations	
	Wed, 11/23/2016		Thanksgiving Holiday -- no class		
	Fri, 11/25/2016		Thanksgiving Holiday -- no class		
35	Mon, 11/28/2016	Chap. 13	Elastic continua	Prepare presentations	
36	Wed, 11/30/2016		Math methods	Prepare presentations	
37	Fri, 12/02/2016		Math methods	Prepare presentations	
38	Mon, 12/05/2016		Math methods	Prepare presentations	
	Wed, 12/07/2016		Presentations I		
	Fri, 12/09/2016		Presentations II		

PHY 711 Presentation Schedule -- Fall 2016

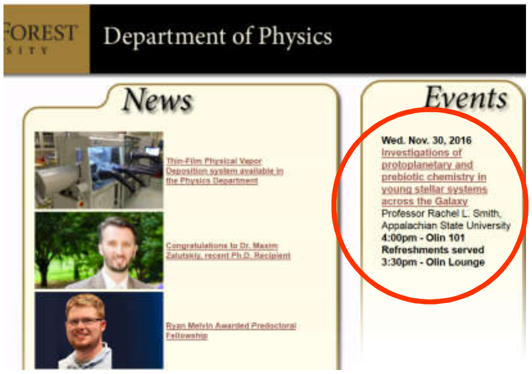
Please enter your name and title next to your prefered time --

Wednesday, Dec. 7, 2016

	Presenter name	Title of presentation
11:00-11:25 AM	Hamna Haneef	Unravelling a classical mechanics brain twister
11:25-11:50 AM	Matt Roveto	Numerical Solutions to the Classical Three Body Problem

Friday, Dec. 9, 2016

	Presenter name	Title of presentation
11:00-11:25 AM	Colin Tyznik	Normal Modes of Perovskite crystal



Arithmetic of complex numbers – from Lecture 19 --

Complex numbers $i \equiv \sqrt{-1}$ $\quad i^2 = -1$

Define $z = x + iy$

$$|z|^2 = zz^* = (x+iy)(x-iy) = x^2 + y^2$$

Polar representation

$$z = \rho(\cos\phi + i\sin\phi) = \rho\, e^{i\phi}$$

Functions of complex variables

$$f(z) = \Re(f(z)) + i\Im(f(z)) \equiv u(x,y) + iv(x,y)$$

Derivatives: Cauchy-Riemann equations

$$\frac{\partial f(z)}{\partial x} = \frac{\partial u(z)}{\partial x} + i\frac{\partial v(z)}{\partial x} \qquad \frac{\partial f(z)}{i\partial y} = \frac{\partial u(z)}{i\partial y} + i\frac{\partial v(z)}{i\partial y} = \frac{\partial v(z)}{\partial y} - i\frac{\partial u(z)}{\partial y}$$

$$\text{Argue that } \frac{df}{dz} = \frac{\partial f(z)}{\partial x} = \frac{\partial f(z)}{i\partial y} \quad \Rightarrow \frac{\partial u(z)}{\partial x} = \frac{\partial v(z)}{\partial y} \quad \text{and} \quad \frac{\partial v(z)}{\partial x} = -\frac{\partial u(z)}{\partial y}$$

Analytic function

$f(z)$ is analytic if it is:

- continuous
- single valued
- its first derivative satisfies Cauchy-Rieman conditions

Which of the following functions are analytic?

$f(z) = e^z$

$f(z) = z^n$

$f(z) = \ln z$

$f(z) = z^\alpha$

Some details

$$e^z = e^{x+iy} = e^x \cos(y) + ie^x \sin(y)$$

$$\frac{\partial u}{\partial x} = e^x \cos(y) = \frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} = e^x \sin(y) = -\frac{\partial u}{\partial y}$$

$$z^2 = \left(x+iy\right)^2 = \left(x^2 - y^2\right) + 2ixy$$

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} = 2y = -\frac{\partial u}{\partial y}$$

More details

$$\ln z = \ln\left(re^{i\theta}\right) = \ln r + i\theta$$

$$= \ln\left(re^{i\theta+i2\pi n}\right) = \ln r + i\theta + i2\pi n \quad \text{for any integer } n$$

$$\Rightarrow \text{ not single-valued}$$

$$z^\alpha = \left(re^{i\theta}\right)^\alpha = r^\alpha e^{i\alpha\theta}$$

$$= \left(re^{i\theta+i2\pi n}\right)^\alpha = r^\alpha e^{i\alpha\theta + i2\pi\alpha n}$$

note that $e^{i2\pi\alpha n} = 1$ only if $\alpha = \text{integer}$

Cauchy integral theorem

For any function $f(z)$ which is analytic within a region enclosed by a closed curve C within the complex plane:

$$\oint_C f(z)dz = 0$$

Example: $f(z) = e^{\alpha z}$ for C a unit circle

$$z = e^{i\theta} \text{ for } 0 \le \theta \le 2\pi$$

$$\oint_C e^{\alpha z}\, dz = \int_0^{2\pi} e^{\alpha(\cos\theta + i\sin\theta)}\, e^{i\theta} i d\theta = 0$$

Special counter-example: $f(z) = \frac{1}{z}$ for C a unit circle

$$\oint_C \frac{dz}{z} = \int_0^{2\pi} \frac{e^{i\theta} i d\theta}{e^{i\theta}} = i\int_0^{2\pi} d\theta = 2\pi i$$

Cauchy's integral representation

For a counterclockwise closed contour C within which function $f(z')$ is analytic:

$$\frac{1}{2\pi i}\oint_C \frac{f(z')}{z'-z}dz' = \begin{cases} f(z) & \text{if } z \text{ is interior to } C \\ 0 & \text{if } z \text{ is exterior to } C \end{cases}$$

Convenient representation of an analytic function:

For a counterclockwise closed contour C on which function $g(z')$ is continuous, $f(z)$ is analytic for z not on C:

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{g(z')}{z'-z}dz'$$

For a counterclockwise closed contour C on which function $g(z')$ is continuous, $f(z)$ is analytic for z not on C:

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{g(z')}{z'-z}dz'$$

It follows that we can evaluate the nth derivative:

$$\frac{d^n f(z)}{dz^n} = \frac{n!}{2\pi i}\oint_C \frac{g(z')}{\left(z'-z\right)^{n+1}}dz'$$

It further follows that $\frac{d^n f(z)}{dz^n}$ is also analytic.

It can be shown that an analytic function $f(z)$ can be represented by a convergent Taylor's series:

$$f(z) = \sum_{n=0}^{\infty} a_n \left(z-z_0\right)^n$$

Special points in complex plane

zero of order n at point z_0 :

A function $f(z)$ has a zero of order n at point z_0 when

$$f(z_0) = 0 = \left.\frac{df(z)}{dz}\right|_{z=z_0} = \ldots = \left.\frac{d^{n-1}f(z)}{dz^{n-1}}\right|_{z=z_0} = 0$$

but $\left.\frac{d^n f(z)}{dz^n}\right|_{z=z_0} \neq 0.$

It follows that near z_0, $f(z) = \left(z-z_0\right)^n h(z)$ where $h(z)$ is non vanishing and analytic.

Isolated singular points of an otherwise analytic function

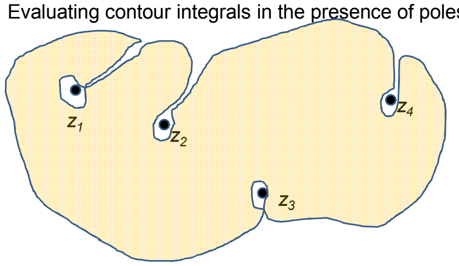
Laurent expansion of function $f(z)$ about a point z_0 :

$$f(z)=\sum_{n=0}^{\infty}a_n\left(z-z_0\right)^n+\frac{b_1}{z-z_0}+\frac{b_2}{\left(z-z_0\right)^2}+...+\frac{b_n}{\left(z-z_0\right)^n}$$

If $b_n \neq 0$ but $b_m = 0$ for $m > n$, $f(z)$ has a pole of order n at $z = z_0$

and $\dfrac{1}{f(z)}$ has a zero of order n at $z = z_0$.

Evaluating contour integrals in the presence of poles



$$\int_C f(z)dz = 2\pi i\sum_{p=1}^{m}\text{Res}\left(f\left(z_p\right)\right)$$

Note that in the vicinity of pole at z_p :

$$f(z)=\sum_{l=0}^{\infty}a_l\left(z-z_p\right)^l+\frac{b_1}{z-z_p}+\frac{b_2}{\left(z-z_p\right)^2}+...+\frac{b_n}{\left(z-z_p\right)^n}$$

$$=\frac{g(z)}{\left(z-z_p\right)^n} \quad \text{where } g(z) \text{ is analytic}$$

$$\text{Res}\left(f(z_p)\right)=\frac{1}{2\pi i}\oint_C\frac{g(z)}{\left(z-z_p\right)^n}dz$$

Note that: $\quad g(z)=\sum_{k=1}^{\infty}a_k\left(z-z_p\right)^k$

$$\Rightarrow\frac{1}{2\pi i}\oint_C\frac{g(z)}{\left(z-z_p\right)^n}dz=\frac{1}{2\pi i}\sum_{k=1}^{\infty}a_k\oint_C\frac{\left(z-z_p\right)^k}{\left(z-z_p\right)^n}dz=a_{n-1}$$

$$=\frac{1}{(n-1)!}\left.\frac{d^{n-1}g(z)}{dz^{n-1}}\right|_{z=z_p}$$

Residue formula:

$$\int_C f(z)dz = 2\pi i \sum_{p=1}^{m} \mathrm{Res}\left(f\left(z_p\right)\right)$$

$$\mathrm{Res}\left(f\left(z_p\right)\right) = \lim_{z \to z_p} \left(\frac{1}{(n-1)!} \left\{ \frac{d^{n-1}}{dz^{n-1}} \left(\left(z - z_p\right)^n f(z) \right) \right\} \right)$$

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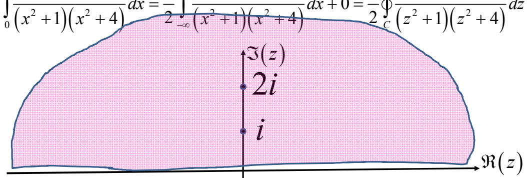
Example integral:

$$I = \int_0^\infty \frac{x^2}{\left(x^2+1\right)\left(x^2+4\right)} dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{x^2}{\left(x^2+1\right)\left(x^2+4\right)} dx + 0 = \frac{1}{2} \oint_C \frac{z^2}{\left(z^2+1\right)\left(z^2+4\right)} dz$$



$$I = 2\pi i \left(\mathrm{Res}\left(f(i)\right) + \mathrm{Res}\left(f(2i)\right) \right)$$

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Example continued:

$$f(z) = \frac{1}{2} \frac{z^2}{\left(z^2+1\right)\left(z^2+4\right)}$$

$$I = 2\pi i \left(\mathrm{Res}\left(f(i)\right) + \mathrm{Res}\left(f(2i)\right) \right)$$

$$\mathrm{Res}\left(f(i)\right) = \frac{1}{2} \frac{i^2}{\left(i+i\right)\left(i^2+4\right)} = \frac{1}{2} \frac{-1}{(2i)(-1+4)} = -\frac{1}{12i}$$

$$\mathrm{Res}\left(f(2i)\right) = \frac{1}{2} \frac{4i^2}{\left(4i^2+1\right)\left(2i+2i\right)} = \frac{1}{2} \frac{-4}{(-4+1)(4i)} = \frac{1}{6i}$$

$$I = \frac{\pi}{6}$$

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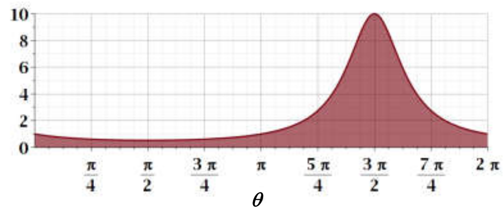
Example:

$$I=\int_0^{2\pi}\frac{d\theta}{1+a\sin\theta}\quad\text{for}\quad 0\le a^2<1$$



$$I=\int_0^{2\pi}\frac{d\theta}{1+a\sin\theta}\quad\text{for}\quad 0\le a^2<1$$

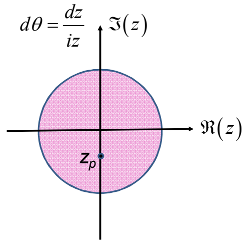
Let $z=e^{i\theta}$ $\quad \sin\theta=\frac{z-1/z}{2i}$ $\quad d\theta=\frac{dz}{iz}$

$$I=\oint_C\frac{2dz}{2iz+az^2-a}$$

Pole within unit circle:

$$z_p=-i\frac{1-\sqrt{1-a^2}}{a}$$

$$I=2\pi i\ \text{Res}\left(f(z_p)\right)=\frac{2\pi}{\sqrt{1-a^2}}$$



Example – Gauss's integral

$$I=\int_{-\infty}^{\infty}e^{iax-bx^2}dx\quad\text{for } a,\ b \text{ real and } b>0.$$

Let $z=x-\frac{ia}{2b}$

$$I=e^{-a^2/4b}\int_{-\infty-ia/2b}^{\infty-ia/2b}e^{-bz^2}dz=e^{-a^2/4b}\int_{-\infty}^{\infty}e^{-bz^2}dz=e^{-a^2/4b}\sqrt{\frac{\pi}{b}}$$

$1+2+3+4=0$

$1=-3 \qquad 2=-4$