PHY 711 Classical Mechanics and Mathematical Methods
11-11:50 AM MWF Olin 107

Plan for Lecture 37

Review of mathematical methods –

- **More comments on complex numbers**
- **Topics concerning matrices**

12/02/2016 PHY 711 Fall 2016 -- Lecture 37 1

23	Wed, 10/26/2016	Chap. 9	Introduction to hydrodynamics		
24	Fri, 10/28/2016	Chap. 9	Introduction to hydrodynamics	#18	10/31/2016
25	Mon, 10/31/2016	Chap. 9	Sound waves	#19	11/02/2016
26	Wed, 11/02/2016	Chap. 9	Sound waves	#20	11/04/2016
27	Fri, 11/04/2016	Chap. 9	Non-linear sound	#21	11/07/2016
28	Mon, 11/07/2016	Chap. 10	Surface waves in fluids		
29	Wed, 11/09/2016	Chap. 10	Surface waves in fluids	#22	11/11/2016
30	Fri, 11/11/2016	Chap. 11	Heat conductivity	#23	11/14/2016
31	Mon, 11/14/2016	Chap. 12	Viscous fluids	#24	11/16/2016
32	Wed, 11/16/2016	Chap. 12	Viscous fluids	#25	11/18/2016
33	Fri, 11/18/2016	Chap. 12	Viscous fluids	#26	11/21/2016
34	Mon, 11/21/2016	Chap. 13	Elastic continua	Prepare presentations	
	Wed, 11/23/2016		Thanksgiving Holiday -- no class		
	Fri, 11/25/2016		Thanksgiving Holiday -- no class		
35	Mon, 11/28/2016	Chap. 13	Elastic continua	Prepare presentations	
36	Wed, 11/30/2016		Math methods	Prepare presentations	
37	Fri, 12/02/2016		Math methods	Prepare presentations	
38	Mon, 12/05/2016		Math methods	Prepare presentations	
	Wed, 12/07/2016		Presentations I		
	Fri, 12/09/2016		Presentations II		

12/02/2016 PHY 711 Fall 2016 -- Lecture 37 2

PHY 711 Presentation Schedule -- Fall 2016

Please enter your name and title next to your prefered time --

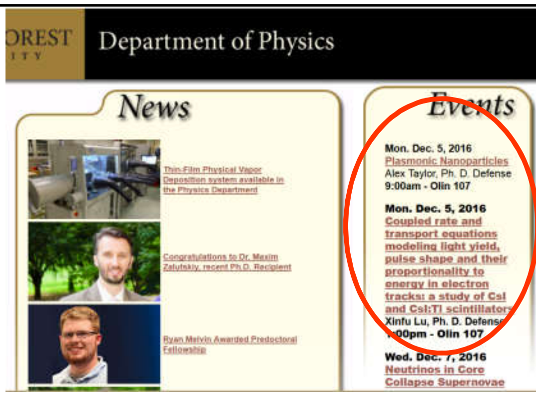
Wednesday, Dec. 7, 2016

	Presenter name	Title of presentation
11:00-11:25 AM	Hamna Haneef	Unravelling a classical mechanics brain twister
11:25-11:50 AM	Matt Roveto	Numerical Solutions to the Classical Three Body Problem

Friday, Dec. 9, 2016

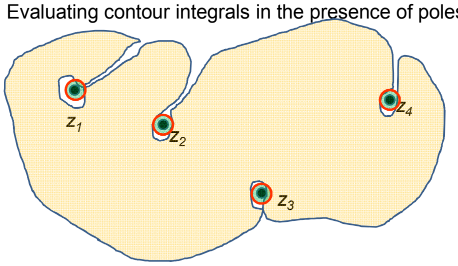
	Presenter name	Title of presentation
11:00-11:25 AM	Colin Tyznik	Normal Modes of Perovskite crystal

12/02/2016 PHY 711 Fall 2016 -- Lecture 37 3



12/02/2016 PHY 711 Fall 2016 -- Lecture 37 4

Evaluating contour integrals in the presence of poles



$$\int_C f(z)dz = 2\pi i \sum_{p=1}^{m} \mathrm{Res}\left(f\left(z_p\right)\right)$$

12/02/2016 PHY 711 Fall 2016 -- Lecture 37 5

A function is “meromorphic” in a region of the complex plane if it is analytic except for poles in that region

It follows that a meromorphic function $f(z)$ can be represented as a ratio of polynomials:

$$f(z) = \frac{a_m z^m + a_{m-1} z^{m-1} + \ldots + a_1 z + a_0}{b_n z^n + b_{n-1} z^{n-1} + \ldots + b_1 z + b_0} \quad \text{where } m \text{ and } n \text{ are integers}$$

12/02/2016 PHY 711 Fall 2016 -- Lecture 37 6

Useful theorem:

A meromorphic function $f(z)$ and an analytic function $g(z)$ are defined in a region R. Consider a closed contour C within R passing through no poles or zeros. Suppose that within C, $f(z)$ has Z zeros at $z = a_1, a_2, \ldots a_Z$ of order $n_1, n_2, \ldots n_Z$ and P poles at $z = b_1, b_2, \ldots b_P$ of order $m_1, m_2, \ldots m_P$.

$$\Rightarrow \frac{1}{2\pi i}\oint_C \frac{g(z)}{f(z)}\frac{df}{dz}dz = \sum_{j=1}^{Z} n_j g(a_j) - \sum_{j=1}^{P} m_j g(b_j)$$

Some details:

Near zero a_j: $f(z) = C(z-a_j)^{n_j}$

$$\frac{g(z)}{f(z)}\frac{df}{dz} = n_j \frac{g(a_j)}{z-a_j}$$

$$\frac{1}{2\pi i}\oint_C \frac{g(z)}{f(z)}\frac{df}{dz}dz = n_j g(a_j)$$

Some details:

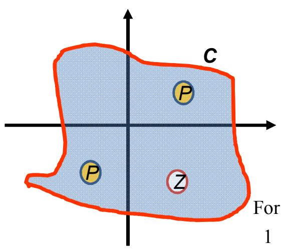
Near pole b_j: $f(z) = C(z-b_j)^{-m_j}$

$$\frac{g(z)}{f(z)}\frac{df}{dz} = -m_j \frac{g(b_j)}{z-b_j}$$

$$\frac{1}{2\pi i}\oint_C \frac{g(z)}{f(z)}\frac{df}{dz}dz = -m_j g(b_j)$$

For $g(z) = 1$:

$$\Rightarrow \frac{1}{2\pi i}\oint_C \frac{1}{f(z)}\frac{df}{dz}dz = \sum_{j=1}^{Z} n_j - \sum_{j=1}^{P} m_j$$



For this example:

$$\frac{1}{2\pi i}\oint_C \frac{1}{f(z)}\frac{df}{dz}dz = 1 - 2 = -1$$

Some useful properties of matrices

$$M = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}$$

It is often necessary to solve linear equations of the form

$$Mx = b$$

which has the formal solution

$$x = M^{-1}b$$

unless M is singular (has vanishing determinant).

Singular value decomposition

$$M = U\Sigma V^T$$

$$\text{where} \quad \Sigma = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix}$$

$$V = \left(\begin{pmatrix} v_{1a} \\ v_{1b} \\ v_{1c} \end{pmatrix} \begin{pmatrix} v_{2a} \\ v_{2b} \\ v_{2c} \end{pmatrix} \begin{pmatrix} v_{3a} \\ v_{3b} \\ v_{3c} \end{pmatrix} \right) \quad \text{orthogonal vectors}$$

$$U = \left(\begin{pmatrix} u_{1a} \\ u_{1b} \\ u_{1c} \end{pmatrix} \begin{pmatrix} u_{2a} \\ u_{2b} \\ u_{2c} \end{pmatrix} \begin{pmatrix} u_{3a} \\ u_{3b} \\ u_{3c} \end{pmatrix} \right) \quad \text{orthogonal vectors}$$

Singular value decomposition -- continued

$$M = U\Sigma V^T$$

$$\text{Note that} \quad \mathbf{v}_i \cdot \mathbf{v}_j = \delta_{ij}$$

$$\text{and that} \quad \mathbf{u}_i \cdot \mathbf{u}_j = \delta_{ij}$$

$$\text{but } \mathbf{u}_i \cdot \mathbf{v}_j \neq \delta_{ij} \quad \text{necessarily}$$

Consider $Mx = b$

$$\text{Suppose:} \quad M = U\Sigma V^T = \sum_{i=1}^{N} \sigma_i u_i v_i^T$$

$$\sum_{i=1}^{N} \sigma_i u_i \left\langle v_i^T \middle| x \right\rangle = b$$

$$\sigma_i \left\langle v_i^T \middle| x \right\rangle = \left\langle u_i^T \middle| b \right\rangle$$

Consider $Mx = b$

$$\text{Suppose:} \quad M = U\Sigma V^T = \sum_{i=1}^{N} \sigma_i u_i v_i^T$$

$$\sum_{i=1}^{N} \sigma_i u_i \left\langle v_i^T \middle| x \right\rangle = b$$

$$\sigma_i \left\langle v_i^T \middle| x \right\rangle = \left\langle u_i^T \middle| b \right\rangle$$

$$\text{If } \sigma_i \neq 0: \quad \left\langle v_i^T \middle| x \right\rangle = \frac{\left\langle u_i^T \middle| b \right\rangle}{\sigma_i}$$

$$\Rightarrow x = \sum_{i=1}^{N} \frac{\left\langle u_i^T \middle| b \right\rangle}{\sigma_i} v_i$$

Example:

$$M = \begin{pmatrix} 0.96 & 1.72 \\ 2.28 & 0.96 \end{pmatrix}$$

$$= U\Sigma V^T = \begin{pmatrix} 0.6 & -0.8 \\ 0.8 & 0.6 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0.8 & 0.6 \\ 0.6 & -0.8 \end{pmatrix}$$

Example:

$$A := \begin{bmatrix} 1 & 1 \\ 1 & 0.9 \end{bmatrix}$$

$$\begin{matrix} \sigma_1 \\ \sigma_2 \end{matrix} = \begin{bmatrix} 1.95124921972504 \\ 0.0512492197250392 \end{bmatrix}$$

$$A1 := \begin{bmatrix} 1 & 1 \\ 1 & 0.99 \end{bmatrix}$$

$$\begin{matrix} \sigma_1 \\ \sigma_2 \end{matrix} = \begin{bmatrix} 1.99501249992188 \\ 0.00501249992187594 \end{bmatrix}$$

Note that:

$$MM^T = U\Sigma V^T \left(U\Sigma V^T\right)^T = U\Sigma V^T V\Sigma U^T = U\Sigma^2 U^T$$

$$M^T M = \left(U\Sigma V^T\right)^T U\Sigma V^T = V\Sigma U^T U\Sigma V^T = V\Sigma^2 V^T$$

$\Rightarrow$ By convention, singular values are chosen as $\sigma_i \geq 0$

Note that SVD can be applied to rectangular matrices