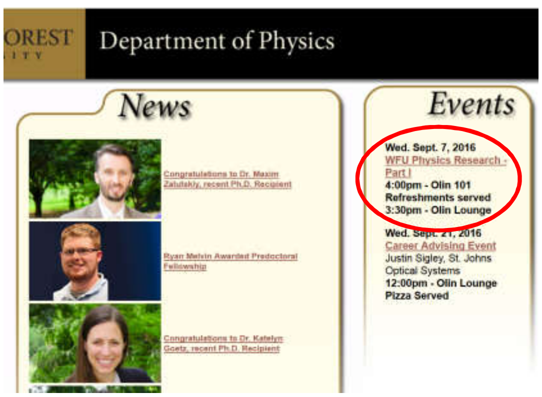
PHY 711 Classical Mechanics and Mathematical Methods
11-11:50 AM MWF Olin 107

Plan for Lecture 3:

1. **Scattering theory in the center of mass frame**
2. **Calculation of the scattering cross section**
3. **Cross section for Rutherford scattering**



WFU Physics Colloquium

TITLE: "WFU Physics Research -- Part I"
TIME: Wednesday Sept. 7, 2016 at 4:00 PM
PLACE: George P. Williams, Jr. Lecture Hall, (Olin 101)

Refreshments will be served at 3:30 PM in the lounge. All interested persons are cordially invited to attend.

PROGRAM

This colloquium is the first of two which will highlight physics research at Wake Forest University. During the colloquium, Physics Department faculty members will present short overviews of their research programs in the Physics Department. This forum for sharing ideas will hopefully inspire collaborations between students and faculty and between research groups.

PHY 711 Classical Mechanics and Mathematical Methods

MWF 11 AM-11:50 AM	OPL 107	http://www.wfu.edu/~natalie/f16phy711/

Instructor: Natalie Holzwarth	Phone:758-5510	Office:300 OPL	e-mail:natalie@wfu.edu

Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W Reading	Topic	Assignment	Due
1	Wed, 8/31/2016	Chap. 1	Review of basic principles	#1	9/7/2016
2	Fri, 9/02/2016	Chap. 1	Scattering theory	#2	9/7/2016
	Mon, 9/05/2016		Labor day -- no class		
3	Wed, 9/07/2016	Chap. 1	Scattering theory	#3	9/9/2016
4	Fri, 9/09/2016	Chap. 1 & 2	Scattering theory and rotations	#4	9/12/2016
5	Mon, 9/12/2016				
6	Wed, 9/14/2016				
7	Fri, 9/16/2016				

PHY 711 -- Assignment #3

Sept. 7, 2016

Continue reading Chapter 1 in **Fetter & Walecka**.

1. Work Problem #1.15 at the end of Chapter 1 in **Fetter and Walecka**.

Scattering theory:

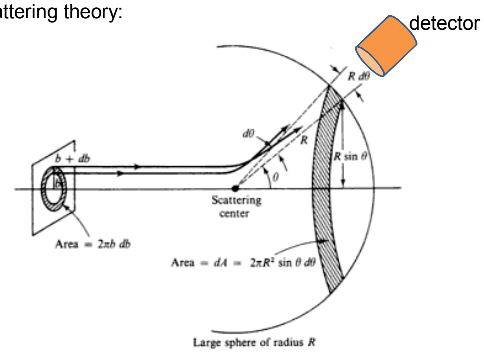
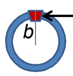


Figure 5.5 The scattering problem and relation of cross section to impact parameter.

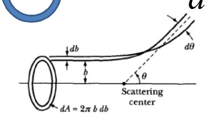
Differential cross section

$$\left(\frac{d\sigma}{d\Omega}\right)=\frac{\text{Number of detected particles at }\theta\text{ per target particle}}{\text{Number of incident particles per unit area}}$$

$=$ Area of incident beam that is scattered into detector at angle θ



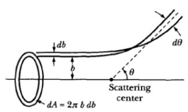
$$d\sigma = d\phi b db$$
$$d\Omega = d\phi \sin\theta d\theta$$

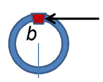


$$\left(\frac{d\sigma}{d\Omega}\right)=\frac{d\varphi\, b\, db}{d\varphi \sin\theta\, d\theta}=\frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

Figure from Marion & Thorton, Classical Dynamics

Note: Notion of cross section is common to many areas of physics including classical mechanics, quantum mechanics, optics, etc. Only in the classical mechanics can we calculate it using geometric considerations





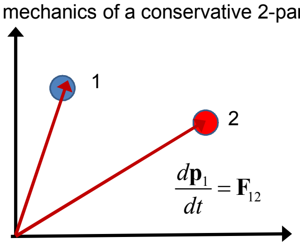
$d\varphi b db$

Figure from Marion & Thorton, Classical Dynamics

$$\left(\frac{d\sigma}{d\Omega}\right)=\frac{d\varphi\, b\, db}{d\varphi \sin\theta\, d\theta}=\frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

Note: We are assuming that the process is isotropic in ϕ

Relationship of scattering cross-section to particle interactions -- Classical mechanics of a conservative 2-particle system.



$$\frac{d\mathbf{p}_1}{dt}=\mathbf{F}_{12} \qquad \frac{d\mathbf{p}_2}{dt}=\mathbf{F}_{21}$$

$$\mathbf{F}_{12}=-\nabla_1 V\left(\mathbf{r}_1-\mathbf{r}_2\right) \quad \Rightarrow E=\frac{1}{2}m_1v_1^2+\frac{1}{2}m_2v_2^2+V\left(\mathbf{r}_1-\mathbf{r}_2\right)$$

Typical two-particle interactions –

Central potential: $V(\mathbf{r}_1-\mathbf{r}_2)=V(|\mathbf{r}_1-\mathbf{r}_2|)\equiv V(r)$

Hard sphere: $V(r)=\begin{cases}\infty & r\le a\\ 0 & r>a\end{cases}$

Coulomb or gravitational: $V(r)=\dfrac{K}{r}$

Lennard-Jones: $V(r)=\dfrac{A}{r^{12}}-\dfrac{B}{r^{6}}$

Relationship between center of mass and laboratory frames of reference

Definition of center of mass $\mathbf{R}_{CM}$

$$m_1\mathbf{r}_1+m_2\mathbf{r}_2=(m_1+m_2)\mathbf{R}_{CM}$$

$$m_1\dot{\mathbf{r}}_1+m_2\dot{\mathbf{r}}_2=(m_1+m_2)\dot{\mathbf{R}}_{CM}\equiv(m_1+m_2)\mathbf{V}_{CM}$$

$$E=\frac{1}{2}m_1v_1^2+\frac{1}{2}m_2v_2^2+V(\mathbf{r}_1-\mathbf{r}_2)$$

$$=\frac{1}{2}(m_1+m_2)V_{CM}^2+\frac{1}{2}\mu|\mathbf{v}_1-\mathbf{v}_2|^2+V(\mathbf{r}_1-\mathbf{r}_2)$$

where: $\mu\equiv\dfrac{m_1m_2}{m_1+m_2}$

Classical mechanics of a conservative 2-particle system -- continued

$$E=\frac{1}{2}(m_1+m_2)V_{CM}^2+\frac{1}{2}\mu|\mathbf{v}_1-\mathbf{v}_2|^2+V(\mathbf{r}_1-\mathbf{r}_2)$$

For central potentials: $V(\mathbf{r}_1-\mathbf{r}_2)=V(|\mathbf{r}_1-\mathbf{r}_2|)\equiv V(r_{12})$

Relative angular momentum is also conserved:

$$\mathbf{L}_{12}\equiv\mathbf{r}_{12}\times\mu\mathbf{v}_{12}$$

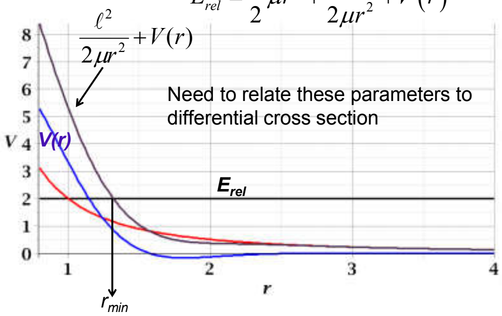
$$E=\frac{1}{2}(m_1+m_2)V_{CM}^2+\frac{1}{2}\mu v_{12}^2+\frac{L_{12}^2}{2\mu r_{12}^2}+V(r_{12})$$

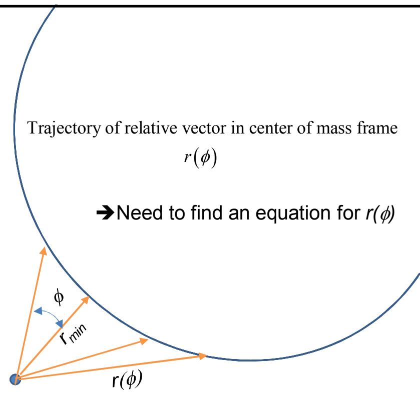
Simpler notation:

$$E=\frac{1}{2}(m_1+m_2)V_{CM}^2+\overbrace{\frac{1}{2}\mu\dot{r}^2+\frac{\ell^2}{2\mu r^2}+V(r)}^{E_{rel}}$$

For a continuous potential interaction in center of mass reference frame:

$$E_{rel} = \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$





Conservation of energy in the center of mass frame:

$$E_{rel} \equiv E = \frac{1}{2}\mu\left(\frac{dr}{dt}\right)^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

Transformation of trajectory variables:

$$r(t) \Leftrightarrow r(\varphi)$$

$$\frac{dr}{dt} = \frac{dr}{d\varphi}\frac{d\varphi}{dt} = \frac{dr}{d\varphi}\frac{\ell}{\mu r^2}$$

Here, constant angular momentum is: $\ell = \mu r^2\left(\frac{d\varphi}{dt}\right)$

$$\Rightarrow E = \frac{1}{2}\mu\left(\frac{dr}{d\varphi}\frac{\ell}{\mu r^2}\right)^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

Solving for $r(\varphi) \Leftrightarrow \varphi(r)$:

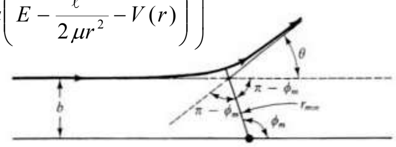
From: $E = \frac{1}{2}\mu\left(\frac{dr}{d\varphi}\frac{\ell}{\mu r^2}\right)^2 + \frac{\ell^2}{2\mu r^2} + V(r)$

$$\left(\frac{dr}{d\varphi}\right)^2 = \left(\frac{2\mu r^4}{\ell^2}\right)\left(E - \frac{\ell^2}{2\mu r^2} - V(r)\right)$$

$$d\varphi = dr\left(\frac{\ell / r^2}{\sqrt{2\mu\left(E - \frac{\ell^2}{2\mu r^2} - V(r)\right)}}\right)$$

$$d\phi = dr\left(\frac{\ell / r^2}{\sqrt{2\mu\left(E - \frac{\ell^2}{2\mu r^2} - V(r)\right)}}\right)$$

v_∞ →



Special values at large separation ($r \to \infty$):

$\ell = \mu v_\infty b$

$E = \frac{1}{2}\mu v_\infty^2$

$\Rightarrow \ell = \sqrt{2\mu E}\, b$

When the dust clears:

$$d\varphi = dr\left(\frac{\ell / r^2}{\sqrt{2\mu\left(E - \frac{\ell^2}{2\mu r^2} - V(r)\right)}}\right)$$

$$d\varphi = dr\left(\frac{b / r^2}{\sqrt{1 - \frac{b^2}{r^2} - \frac{V(r)}{E}}}\right)$$

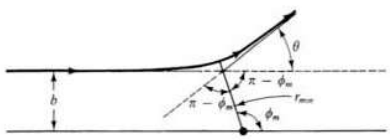
$$\Rightarrow \varphi_{\max}(b, E) = \varphi(r \to \infty) - \varphi(r = r_{\min})$$

$$\int_0^{\phi_{max}} d\phi = \int_{r_{min}}^{\infty} dr \left(\frac{b/r^2}{\sqrt{1-\frac{b^2}{r^2}-\frac{V(r)}{E}}} \right)$$

where :

$$1-\frac{b^2}{r_{min}^2}-\frac{V(r_{min})}{E}=0$$

Relationship between ϕ_{max} and θ:



$$2(\pi-\varphi_{max})+\theta=\pi$$

$$\Rightarrow \varphi_{max}=\frac{\pi}{2}+\frac{\theta}{2}$$

$$\varphi_{max}=\frac{\pi}{2}+\frac{\theta}{2}=\int_{r_{min}}^{\infty} dr \left(\frac{b/r^2}{\sqrt{1-\frac{b^2}{r^2}-\frac{V(r)}{E}}} \right)$$

$$\theta=-\pi+2b\int_{r_{min}}^{\infty} dr \left(\frac{1/r^2}{\sqrt{1-\frac{b^2}{r^2}-\frac{V(r)}{E}}} \right)$$

$$\theta=-\pi+2b\int_0^{1/r_{min}} du \left(\frac{1}{\sqrt{1-b^2u^2-\frac{V(1/u)}{E}}} \right)$$

Scattering angle equation:

$$\theta = -\pi + 2b \int_0^{1/r_{\min}} du \left(\frac{1}{\sqrt{1 - b^2u^2 - \frac{V(1/u)}{E}}} \right)$$

where:

$$1 - \frac{b^2}{r_{\min}^2} - \frac{V(r_{\min})}{E} = 0$$

Rutherford scattering example:

$$\frac{V(r)}{E} \equiv \frac{\kappa}{r} \qquad 1 - \frac{b^2}{r_{\min}^2} - \frac{\kappa}{r_{\min}} = 0$$

$$\frac{1}{r_{\min}} = \frac{1}{b}\left(-\frac{\kappa}{2b} + \sqrt{\left(\frac{\kappa}{2b}\right)^2 + 1} \right)$$

$$\theta = -\pi + 2b \int_0^{1/r_{\min}} du \left(\frac{1}{\sqrt{1 - b^2u^2 - \kappa u}} \right) = 2\sin^{-1}\left(\frac{1}{\sqrt{(2b/\kappa)^2 + 1}} \right)$$

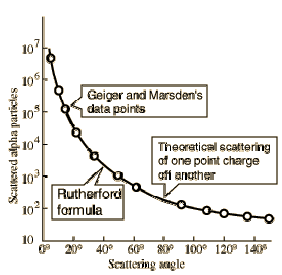
Rutherford scattering continued :

$$\theta = 2\sin^{-1}\left(\frac{1}{\sqrt{(2b/\kappa)^2 + 1}} \right)$$

$$\frac{2b}{\kappa} = \left| \frac{\cos(\theta/2)}{\sin(\theta/2)} \right|$$

$$\left(\frac{d\sigma}{d\Omega} \right) = \frac{b}{\sin\theta} \left| \frac{db}{d\theta} \right| = \frac{\kappa^2}{16} \frac{1}{\sin^4(\theta/2)}$$

$$\left(\frac{d\sigma}{d\Omega} \right) = \frac{b}{\sin\theta} \left| \frac{db}{d\theta} \right| = \frac{\kappa^2}{16} \frac{1}{\sin^4(\theta/2)}$$



What happens as $\theta \rightarrow 0$?

From webpage: http://hyperphysics.phy-astr.gsu.edu/hbase/nuclear/rutsca2.html#c3

Original experiment performed with α particles on gold

$$\frac{\kappa}{4} = \frac{Z_\alpha Z_{\mathrm{Au}} e^2}{8\pi\epsilon_0 \mu v_\infty^2}$$