

**PHY 711 Classical Mechanics and
Mathematical Methods
11-11:50 AM MWF Olin 107**

Plan for Lecture 4:

1. Chap. 1 -- Summary of results from scattering analysis
2. Chap. 2 – Physics described in an accelerated coordinate frame
 - a. Linear acceleration
 - b. Angular acceleration
 - c. Foucault pendulum

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PHY 711 Classical Mechanics and Mathematical Methods

MWF 11 AM-11:50 AM | OPL 107 | <http://www.wfu.edu/~natalie/f16phy711/>

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Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

Date	F&W Reading	Topic	Assignment Due
1 Wed, 8/31/2016	Chap. 1	Review of basic principles	#1 9/7/2016
2 Fri, 9/02/2016	Chap. 1	Scattering theory	#2 9/7/2016
Mon, 9/05/2016		Labor day -- no class	
3 Wed, 9/07/2016	Chap. 1	Scattering theory	#3 9/9/2016
4 Fri, 9/09/2016	Chap. 1 & 2	Scattering theory and rotations	#4 9/12/2016
5 Mon, 9/12/2016			
6 Wed, 9/14/2016			
7 Fri, 9/16/2016			



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PHY 711 -- Assignment #4

Sept. 9, 2016

Finish reading Chapter 1 and start reading Chapter 2 in **Fetter & Walecka**.

1. Work Problem #1.16 at the end of Chapter 1 in **Fetter and Walecka**.

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Summary of scattering analysis

Figure 5.5 The scattering problem and relation of cross section to impact parameter.

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Differential cross section

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{\text{Number of detected particles at } \theta \text{ per target particle}}{\text{Number of incident particles per unit area}}$$

= Area of incident beam that is scattered into detector at angle θ

$$d\sigma = d\phi b db$$

$$d\Omega = d\phi \sin \theta d\theta$$

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{d\phi b db}{d\phi \sin \theta d\theta} = \frac{b}{\sin \theta} \left|\frac{db}{d\theta}\right|$$

Figure from Marion & Thornton, Classical Dynamics

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Transformation between center-of-mass and laboratory reference frames:

Lab CM

$$\mathbf{v}_1 = \mathbf{V}_1 + \mathbf{V}_{CM}$$

$$\tan \psi = \frac{\sin \theta}{\cos \theta + m_1 / m_2}$$

$$\cos \psi = \frac{\cos \theta + m_1 / m_2}{\sqrt{1 + 2m_1 / m_2 \cos \theta + (m_1 / m_2)^2}}$$

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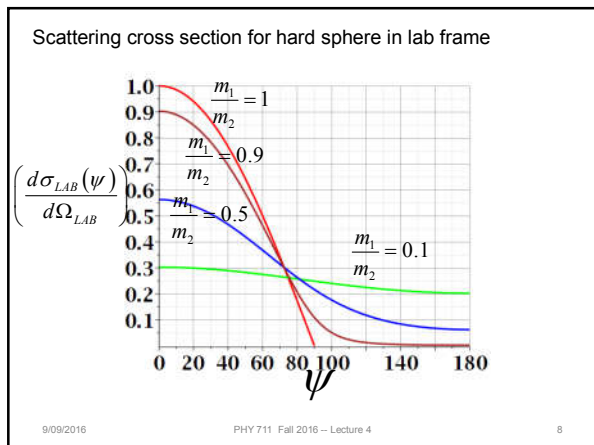
$$\left(\frac{d\sigma_{LAB}(\psi)}{d\Omega_{LAB}} \right) = \left(\frac{d\sigma_{CM}(\theta)}{d\Omega_{CM}} \right) \left| \frac{d \cos \theta}{d \cos \psi} \right|$$

$$\left(\frac{d\sigma_{LAB}(\psi)}{d\Omega_{LAB}} \right) = \left(\frac{d\sigma_{CM}(\theta)}{d\Omega_{CM}} \right) \frac{(1 + 2m_1/m_2 \cos \theta + (m_1/m_2)^2)^{3/2}}{(m_1/m_2) \cos \theta + 1}$$

It is convenient to make a parametric plot of

$$\left(\frac{\sigma_{LAB}(\theta)}{\Omega} \right) \text{ vs } \psi(\theta)$$

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Physics in accelerated reference frames

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Physical laws as described in non-inertial coordinate systems

- Newton's laws are formulated in an inertial frame of reference $\{\hat{e}_i^0\}$
- For some problems, it is convenient to transform the equations into a non-inertial coordinate system $\{\hat{e}_i(t)\}$

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Comparison of analysis in "inertial frame" versus "non-inertial frame"

Denote by \hat{e}_i^0 a fixed coordinate system
 Denote by \hat{e}_i a moving coordinate system

$$\mathbf{V} = \sum_{i=1}^3 V_i^0 \hat{e}_i^0 = \sum_{i=1}^3 V_i \hat{e}_i$$

$$\left(\frac{d\mathbf{V}}{dt}\right)_{inertial} = \sum_{i=1}^3 \frac{dV_i^0}{dt} \hat{e}_i^0 = \sum_{i=1}^3 \frac{dV_i}{dt} \hat{e}_i + \sum_{i=1}^3 V_i \frac{d\hat{e}_i}{dt}$$

Define: $\left(\frac{d\mathbf{V}}{dt}\right)_{body} \equiv \sum_{i=1}^3 \frac{dV_i}{dt} \hat{e}_i$

$$\Rightarrow \left(\frac{d\mathbf{V}}{dt}\right)_{inertial} = \left(\frac{d\mathbf{V}}{dt}\right)_{body} + \sum_{i=1}^3 V_i \frac{d\hat{e}_i}{dt}$$

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Properties of the frame motion (rotation only):

Here $d\Omega \equiv d\Omega \hat{x}$
 $\omega \equiv \frac{d\Omega}{dt} \hat{x}$

$$d\hat{e}_y = d\Omega \hat{e}_z$$

$$d\hat{e}_z = -d\Omega \hat{e}_y$$

$$\Rightarrow d\hat{e} = d\Omega \times \hat{e}$$

$$\frac{d\hat{e}}{dt} = \frac{d\Omega}{dt} \times \hat{e}$$

$$\frac{d\hat{e}}{dt} = \omega \times \hat{e}$$

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Properties of the frame motion (rotation only):

$$d\hat{\mathbf{e}} = d\Omega \times \hat{\mathbf{e}} \quad \frac{d\hat{\mathbf{e}}}{dt} = \frac{d\Omega}{dt} \times \hat{\mathbf{e}} \quad \frac{d\hat{\mathbf{e}}}{dt} = \boldsymbol{\omega} \times \hat{\mathbf{e}}$$

Rotation about x-axis:

$$\begin{pmatrix} e_y \\ e_z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e_y \\ e_z \end{pmatrix} \quad \begin{pmatrix} e_y + de_y \\ e_z + de_z \end{pmatrix} = \begin{pmatrix} \cos(d\Omega) & -\sin(d\Omega) \\ \sin(d\Omega) & \cos(d\Omega) \end{pmatrix} \begin{pmatrix} e_y \\ e_z \end{pmatrix}$$

$$\begin{pmatrix} de_y \\ de_z \end{pmatrix} = \begin{pmatrix} \cos(d\Omega)-1 & -\sin(d\Omega) \\ \sin(d\Omega) & \cos(d\Omega)-1 \end{pmatrix} \begin{pmatrix} e_y \\ e_z \end{pmatrix} \approx \begin{pmatrix} 0 & -d\Omega \\ d\Omega & 0 \end{pmatrix} \begin{pmatrix} e_y \\ e_z \end{pmatrix}$$

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Properties of the frame motion (rotation only):

$$d\hat{\mathbf{e}} = d\Omega \times \hat{\mathbf{e}} \quad \frac{d\hat{\mathbf{e}}}{dt} = \frac{d\Omega}{dt} \times \hat{\mathbf{e}} \quad \frac{d\hat{\mathbf{e}}}{dt} = \boldsymbol{\omega} \times \hat{\mathbf{e}}$$

Rotation about x-axis:

$$\begin{pmatrix} de_y \\ de_z \end{pmatrix} \approx \begin{pmatrix} 0 & -d\Omega \\ d\Omega & 0 \end{pmatrix} \begin{pmatrix} e_y \\ e_z \end{pmatrix} = d\Omega \hat{\mathbf{e}}_y - d\Omega \hat{\mathbf{e}}_z = d\Omega \hat{\mathbf{x}} \times \hat{\mathbf{e}}$$

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Properties of the frame motion (rotation only) -- continued

$$\left(\frac{d\mathbf{V}}{dt} \right)_{inertial} = \left(\frac{d\mathbf{V}}{dt} \right)_{body} + \sum_{i=1}^3 V_i \frac{d\hat{\mathbf{e}}_i}{dt}$$

$$\left(\frac{d\mathbf{V}}{dt} \right)_{inertial} = \left(\frac{d\mathbf{V}}{dt} \right)_{body} + \boldsymbol{\omega} \times \mathbf{V}$$

Effects on acceleration (rotation only):

$$\left(\frac{d}{dt} \frac{d\mathbf{V}}{dt} \right)_{inertial} = \left(\left(\frac{d}{dt} \right)_{body} + \boldsymbol{\omega} \times \right) \left\{ \left(\frac{d\mathbf{V}}{dt} \right)_{body} + \boldsymbol{\omega} \times \mathbf{V} \right\}$$

$$\left(\frac{d^2\mathbf{V}}{dt^2} \right)_{inertial} = \left(\frac{d^2\mathbf{V}}{dt^2} \right)_{body} + 2\boldsymbol{\omega} \times \left(\frac{d\mathbf{V}}{dt} \right)_{body} + \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{V} + \boldsymbol{\omega} \times \boldsymbol{\omega} \times \mathbf{V}$$

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Application of Newton's laws in a coordinate system which has an angular velocity $\boldsymbol{\omega}$ and linear acceleration $\left(\frac{d^2\mathbf{a}}{dt^2}\right)_{inertial}$

Newton's laws; Let \mathbf{r} denote the position of particle of mass m :

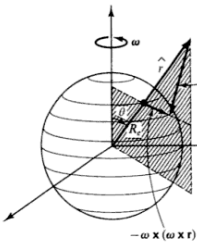
$$m\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{inertial} = \mathbf{F}_{ext}$$

$$m\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{body} = \mathbf{F}_{ext} - m\left(\frac{d^2\mathbf{a}}{dt^2}\right)_{inertial} - 2m\boldsymbol{\omega} \times \left(\frac{d\mathbf{r}}{dt}\right)_{body} - m\frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r} - m\boldsymbol{\omega} \times \boldsymbol{\omega} \times \mathbf{r}$$

↑
↑
 Coriolis force Centrifugal force

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Motion on the surface of the Earth:



$$\omega = \frac{2\pi}{\tau} \approx 7.3 \times 10^{-5} \text{ rad/s}$$

$$\mathbf{F}_{ext} = -\frac{GM_e m}{r^2} \hat{\mathbf{r}} + \mathbf{F}'$$

↑
↑
 Earth's gravity Support force

Main contributions:

$$m\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{earth} = -\frac{GM_e m}{r^2} \hat{\mathbf{r}} + \mathbf{F}' - 2m\boldsymbol{\omega} \times \left(\frac{d\mathbf{r}}{dt}\right)_{earth} - m\frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r} - m\boldsymbol{\omega} \times \boldsymbol{\omega} \times \mathbf{r}$$

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Non-inertial effects on effective gravitational "constant"

$$m\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{earth} = -\frac{GM_e m}{r^2} \hat{\mathbf{r}} + \mathbf{F}' - 2m\boldsymbol{\omega} \times \left(\frac{d\mathbf{r}}{dt}\right)_{earth} - m\frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r} - m\boldsymbol{\omega} \times \boldsymbol{\omega} \times \mathbf{r}$$

For $\left(\frac{d\mathbf{r}}{dt}\right)_{earth} = 0$ and $\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{earth} = 0$,

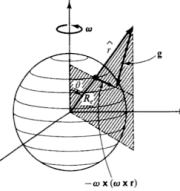
$$0 = -\frac{GM_e m}{r^2} \hat{\mathbf{r}} + \mathbf{F}' - m\boldsymbol{\omega} \times \boldsymbol{\omega} \times \mathbf{r}$$

$$\mathbf{F}' = -m\mathbf{g}$$

$$\Rightarrow \mathbf{g} = -\frac{GM_e}{r^2} \hat{\mathbf{r}} - \boldsymbol{\omega} \times \boldsymbol{\omega} \times \mathbf{r} \Big|_{r=R_e}$$

$$= \left(-\frac{GM_e}{R_e^2} + \omega^2 R_e \sin^2 \theta \right) \hat{\mathbf{r}} + \sin \theta \cos \theta \omega^2 R_e \hat{\boldsymbol{\theta}}$$

↑
↑
 9.80 m/s² 0.03 m/s²



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Foucault pendulum http://www.si.edu/Encyclopedia_SI/nmah/pendulum.htm



The Foucault pendulum was displayed for many years in the Smithsonian's National Museum of American History. It is named for the French physicist Jean Foucault who first used it in 1851 to demonstrate the rotation of the earth.

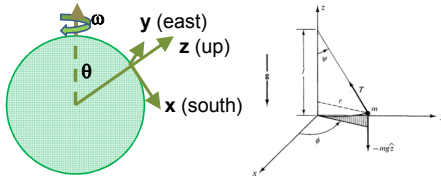
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Equation of motion on Earth's surface

$$m \left(\frac{d^2 \mathbf{r}}{dt^2} \right)_{earth} = -\frac{GM_e m}{r^2} \hat{\mathbf{r}} + \mathbf{F}' - 2m\boldsymbol{\omega} \times \left(\frac{d\mathbf{r}}{dt} \right)_{earth} - m \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r} - m\boldsymbol{\omega} \times \boldsymbol{\omega} \times \mathbf{r}$$



$$\boldsymbol{\omega} \approx -\omega \sin \theta \hat{\mathbf{x}} + \omega \cos \theta \hat{\mathbf{z}}$$

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Foucault pendulum continued – keeping leading terms:

$$m \left(\frac{d^2 \mathbf{r}}{dt^2} \right)_{earth} \approx -\frac{GM_e m}{R_e^2} \hat{\mathbf{r}} + \mathbf{F}' - 2m\boldsymbol{\omega} \times \left(\frac{d\mathbf{r}}{dt} \right)_{earth}$$

$$-\frac{GM_e m}{r^2} \hat{\mathbf{r}} \approx -mg\hat{\mathbf{z}}$$

$$\mathbf{F}' \approx -T \sin \psi \cos \phi \hat{\mathbf{x}} - T \sin \psi \sin \phi \hat{\mathbf{y}} + T \cos \psi \hat{\mathbf{z}}$$

$$\boldsymbol{\omega} \approx -\omega \sin \theta \hat{\mathbf{x}} + \omega \cos \theta \hat{\mathbf{z}}$$

$$\boldsymbol{\omega} \times \left(\frac{d\mathbf{r}}{dt} \right)_{earth} \approx \omega (-\dot{y} \cos \theta \hat{\mathbf{x}} + (\dot{x} \cos \theta + \dot{z} \sin \theta) \hat{\mathbf{y}} - \dot{y} \sin \theta \hat{\mathbf{z}})$$

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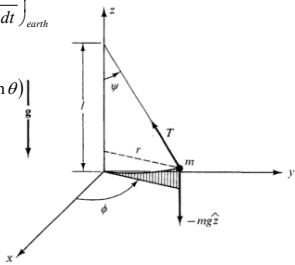
Foucault pendulum continued – keeping leading terms:

$$m \left(\frac{d^2 \mathbf{r}}{dt^2} \right)_{\text{earth}} \approx -\frac{GM_e m}{R_e^2} \hat{\mathbf{r}} + \mathbf{F}' - 2m\boldsymbol{\omega} \times \left(\frac{d\mathbf{r}}{dt} \right)_{\text{earth}}$$

$$\left. \begin{aligned} m\ddot{x} &\approx -T \sin \psi \cos \phi + 2m\omega \dot{y} \cos \theta \\ m\ddot{y} &\approx -T \sin \psi \sin \phi - 2m\omega(\dot{x} \cos \theta + \dot{z} \sin \theta) \\ m\ddot{z} &\approx T \cos \psi - mg + 2m\omega \dot{y} \sin \theta \end{aligned} \right\}$$

Further approximation :
 $\psi \ll 1; \quad \dot{z} \approx 0; \quad T \approx mg$
 $m\ddot{x} \approx -mg \sin \psi \cos \phi + 2m\omega \dot{y} \cos \theta$
 $m\ddot{y} \approx -mg \sin \psi \sin \phi - 2m\omega \dot{x} \cos \theta$

Also note that :
 $x \approx \ell \sin \psi \cos \phi$
 $y \approx \ell \sin \psi \sin \phi$



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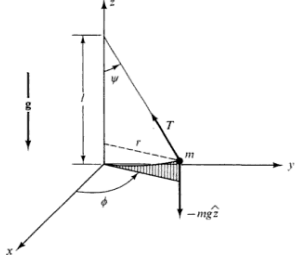
Foucault pendulum continued – coupled equations:

$$\ddot{x} \approx -\frac{g}{\ell} x + 2\omega \cos \theta \dot{y}$$

$$\ddot{y} \approx -\frac{g}{\ell} y - 2\omega \cos \theta \dot{x}$$

Try to find a solution of the form:
 $x(t) = X e^{-iqt} \quad y(t) = Y e^{-iqt}$
 Denote $\omega_{\perp} \equiv \omega \cos \theta$
 $\begin{pmatrix} -q^2 + \frac{g}{\ell} & i2\omega_{\perp} q \\ -i2\omega_{\perp} q & -q^2 + \frac{g}{\ell} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = 0$

Non - trivial solutions :
 $q_{\pm} = \alpha \pm \beta \equiv \omega_{\perp} \pm \sqrt{\omega_{\perp}^2 + \frac{g}{\ell}}$



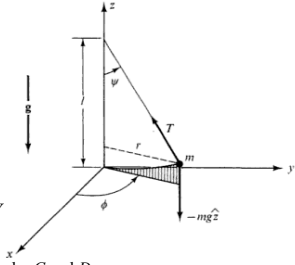
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Foucault pendulum continued – coupled equations:

Solution continued :
 $x(t) = X e^{-iqt} \quad y(t) = Y e^{-iqt}$
 $\begin{pmatrix} -q^2 + \frac{g}{\ell} & i2\omega_{\perp} q \\ -i2\omega_{\perp} q & -q^2 + \frac{g}{\ell} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = 0$

Non - trivial solutions :
 $q_{\pm} = \alpha \pm \beta \equiv \omega_{\perp} \pm \sqrt{\omega_{\perp}^2 + \frac{g}{\ell}}$
 Amplitude relationship : $X = iY$

General solution with complex amplitudes C and D :
 $x(t) = \text{Re} \{ C e^{-(\alpha+\beta)t} + i D e^{-(\alpha-\beta)t} \}$
 $y(t) = \text{Re} \{ C e^{-(\alpha+\beta)t} + D e^{-(\alpha-\beta)t} \}$



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General solution with complex amplitudes C and D :

$$x(t) = \text{Re}\{iCe^{-i(\alpha+\beta)t} + iDe^{-i(\alpha-\beta)t}\}$$

$$y(t) = \text{Re}\{Ce^{-i(\alpha+\beta)t} + De^{-i(\alpha-\beta)t}\}$$

$$q_{\pm} = \alpha \pm \beta \equiv \omega_{\perp} \pm \sqrt{\omega_{\perp}^2 + \frac{g}{\ell}} \approx \omega_{\perp} \pm \sqrt{\frac{g}{\ell}}$$

since $\omega_{\perp} \approx 7 \times 10^{-5} \cos \theta \text{ rad/s} \ll \sqrt{\frac{g}{\ell}}$

Suppose: $x(0) = X_0 \quad y(0) = 0$

$$x(t) = X_0 \cos\left(\sqrt{\frac{g}{\ell}}t\right) \cos(\omega_{\perp}t)$$

$$y(t) = -X_0 \cos\left(\sqrt{\frac{g}{\ell}}t\right) \sin(\omega_{\perp}t)$$

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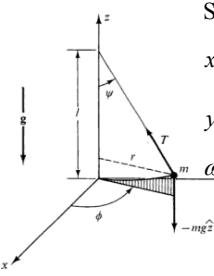
Summary of approximate solution for Foucault pendulum:

Suppose: $x(0) = X_0 \quad y(0) = 0$

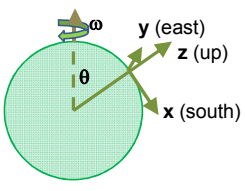
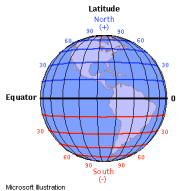
$$x(t) = X_0 \cos\left(\sqrt{\frac{g}{\ell}}t\right) \cos(\omega_{\perp}t)$$

$$y(t) = -X_0 \cos\left(\sqrt{\frac{g}{\ell}}t\right) \sin(\omega_{\perp}t)$$

$\omega_{\perp} \equiv \omega_0 \cos \theta$



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$\omega_{\perp} \equiv \omega_0 \cos \theta$

$$x(t) = X_0 \cos\left(\sqrt{\frac{g}{\ell}}t\right) \cos(\omega_{\perp}t)$$

$$y(t) = -X_0 \cos\left(\sqrt{\frac{g}{\ell}}t\right) \sin(\omega_{\perp}t)$$

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