PHY 711 Classical Mechanics and Mathematical Methods
11-11:50 AM MWF Olin 107

Plan for Lecture 5:

Start reading Chapter 3.17 –

1. **Introduction to the calculus of variations**
2. **Example problems**
3. **Brachistochrone**

PHY 711 Classical Mechanics and Mathematical Methods

MWF 11 AM-11:50 AM	OPL 107	http://www.wfu.edu/~natalie/f16phy711/

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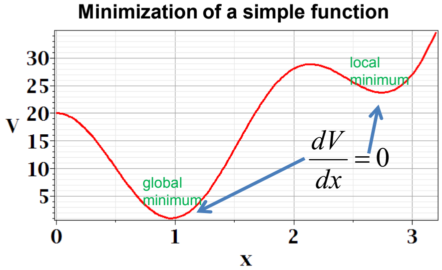
Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W Reading	Topic	Assignment	Due
1	Wed, 8/31/2016	Chap. 1	Review of basic principles	#1	9/7/2016
2	Fri, 9/02/2016	Chap. 1	Scattering theory	#2	9/7/2016
	Mon, 9/05/2016		Labor day -- no class		
3	Wed, 9/07/2016	Chap. 1	Scattering theory	#3	9/9/2016
4	Fri, 9/09/2016	Chap. 1 & 2	Scattering theory and rotations	#4	9/12/2016
5	Mon, 9/12/2016	Chap. 3	Calculus of variations	#5	9/14/2016
6	Wed, 9/14/2016				
7	Fri, 9/16/2016				
8	Mon, 9/19/2016				

In Chapter 3, the notion of Lagrangian dynamics is developed; reformulating Newton’s laws in terms of minimization of related functions. In preparation, we need to develop a mathematical tool known as “the calculus of variation”.

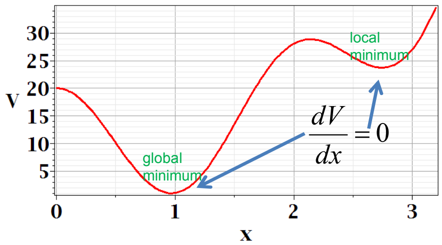
Minimization of a simple function



Minimization of a simple function

Given a function $V(x)$, find the value(s) of x for which $V(x)$ is minimized (or maximized).

Necessary condition : $\frac{dV}{dx} = 0$



Functional minimization

Consider a family of functions $y(x)$, with the end points

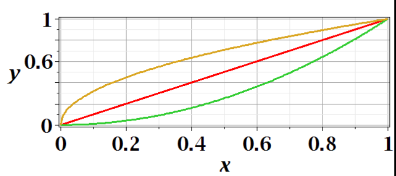
$y(x_i) = y_i$ and $y(x_f) = y_f$ and a function $L\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right)$.

Find the function $y(x)$ which extremizes $L\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right)$.

Necessary condition : $\delta L = 0$

Example :

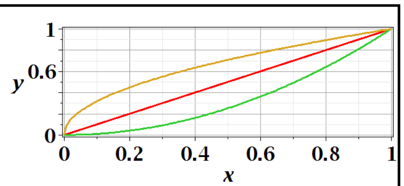
$$L = \int_{(0,0)}^{1,1} \sqrt{(dx)^2 + (dy)^2}$$



Example :

$$L = \int_{(0,0)}^{1,1} \sqrt{(dx)^2 + (dy)^2}$$

$$= \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$



Sample functions :

$y_1(x) = \sqrt{x}$ $\quad L = \int_0^1 \sqrt{1 + \frac{1}{4x}}\, dx = 1.4789$

$y_2(x) = x$ $\quad L = \int_0^1 \sqrt{1+1}\, dx = \sqrt{2} = 1.4142$

$y_2(x) = x^2$ $\quad L = \int_0^1 \sqrt{1 + 4x^2}\, dx = 1.4789$

Calculus of variation example for a pure integral functions

Find the function $y(x)$ which extremizes $L\left(\left\{y(x),\frac{dy}{dx}\right\},x\right)$

where $L\left(\left\{y(x),\frac{dy}{dx}\right\},x\right) \equiv \int_{x_i}^{x_f} f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right)dx.$

Necessary condition : $\delta L = 0$

At any x, let $y(x) \rightarrow y(x) + \delta y(x)$

$$\frac{dy(x)}{dx} \rightarrow \frac{dy(x)}{dx} + \delta\frac{dy(x)}{dx}$$

Formally :

$$\delta L = \int_{x_i}^{x_f}\left[\left(\frac{\partial f}{\partial y}\right)_{x,\frac{dy}{dx}}\delta y + \left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\delta\left(\frac{dy}{dx}\right)\right]\right]dx.$$

After some derivations, we find

$$\delta L = \int_{x_i}^{x_f}\left[\left(\frac{\partial f}{\partial y}\right)_{x,\frac{dy}{dx}}\delta y + \left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\delta\left(\frac{dy}{dx}\right)\right]\right]dx$$

$$= \int_{x_i}^{x_f}\left[\left(\frac{\partial f}{\partial y}\right)_{x,\frac{dy}{dx}} - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\right]\right]\delta y dx = 0 \quad \text{for all } x_i \le x \le x_f$$

$$\Rightarrow \left(\frac{\partial f}{\partial y}\right)_{x,\frac{dy}{dx}} - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\right] = 0 \quad \text{for all } x_i \le x \le x_f$$

Example : End points -- $y(0) = 0; y(1) = 1$

$$L = \int_0^1 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \quad \Rightarrow f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right) = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$\left(\frac{\partial f}{\partial y}\right)_{x,\frac{dy}{dx}} - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\right] = 0$$

$$\Rightarrow -\frac{d}{dx}\left(\frac{dy/dx}{\sqrt{1+(dy/dx)^2}}\right) = 0$$

Solution:

$$\left(\frac{dy/dx}{\sqrt{1+(dy/dx)^2}}\right) = K \qquad \frac{dy}{dx} = K' \equiv \frac{K}{\sqrt{1-K^2}}$$

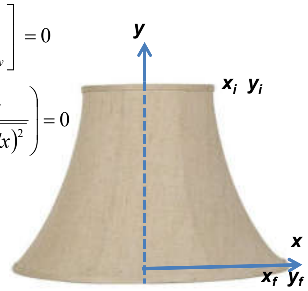
$$\Rightarrow y(x) = K'x + C \qquad y(x) = x$$

Example : Lamp shade shape *y(x)*

$$A = 2\pi \int_{x_i}^{x_f} x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \quad \Rightarrow f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right) = x\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$\left(\frac{\partial f}{\partial y}\right)_{x,\frac{dy}{dx}} - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\right] = 0$$

$$\Rightarrow -\frac{d}{dx}\left(\frac{x dy/dx}{\sqrt{1+(dy/dx)^2}}\right) = 0$$

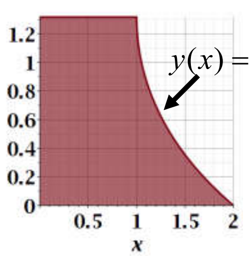


$$-\frac{d}{dx}\left(\frac{x dy/dx}{\sqrt{1+(dy/dx)^2}}\right) = 0$$

$$\frac{x dy/dx}{\sqrt{1+(dy/dx)^2}} = K_1$$

$$\frac{dy}{dx} = -\frac{1}{\sqrt{\left(\frac{x}{K_1}\right)^2 - 1}}$$

$$\Rightarrow y(x) = K_2 - K_1 \ln\left(\frac{x}{K_1} + \sqrt{\frac{x^2}{K_1^2} - 1}\right)$$



$$y(x) = \ln\left(\frac{2+\sqrt{3}}{x+\sqrt{x^2-1}}\right)$$

Another example:
(Courtesy of F. B. Hildebrand, Methods of Applied Mathematics)

Consider all curves $y(x)$ with $y(0)=0$ and $y(1)=1$ that minimize the integral:

$$I=\int_0^1\left(\left(\frac{dy}{dx}\right)^2-ay^2\right)dx \quad \text{for constant } a>0$$

Euler - Lagrange equation:

$$\frac{d^2y}{dx^2}+ay=0$$

$$\Rightarrow y(x)=\frac{\sin(\sqrt{a}x)}{\sin(\sqrt{a})}$$

Review: for $f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right)$,

a necessary condition to extremize $\int_{x_i}^{x_f} f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right)dx$:

$$\left(\frac{\partial f}{\partial y}\right)_{x,\frac{dy}{dx}}-\frac{d}{dx}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\right]=0$$

⇦ Euler-Lagrange equation

Note that for $f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right)$,

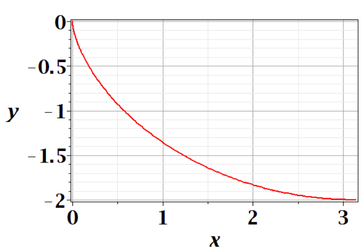
$$\frac{df}{dx}=\left(\frac{\partial f}{\partial y}\right)\frac{dy}{dx}+\left(\frac{\partial f}{\partial(dy/dx)}\right)\frac{d}{dx}\frac{dy}{dx}+\left(\frac{\partial f}{\partial x}\right)$$

$$=\left(\frac{d}{dx}\left(\frac{\partial f}{\partial(dy/dx)}\right)\right)\frac{dy}{dx}+\left(\frac{\partial f}{\partial(dy/dx)}\right)\frac{d}{dx}\frac{dy}{dx}+\left(\frac{\partial f}{\partial x}\right)$$

$$\Rightarrow \frac{d}{dx}\left(f-\frac{\partial f}{\partial(dy/dx)}\frac{dy}{dx}\right)=\left(\frac{\partial f}{\partial x}\right)$$

⇦ Alternate Euler-Lagrange equation

Brachistochrone problem: (solved by Newton in 1696)
http://mathworld.wolfram.com/BrachistochroneProblem.html



A particle of weight *mg* travels frictionlessly down a path of shape *y(x)*. What is the shape of the path *y(x)* that minimizes the travel time from *y(0)=0* to *y(π)=-2*?

$$T=\int_{x_i y_i}^{x_f y_f}\frac{ds}{v}=\int_{x_i}^{x_f}\frac{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\sqrt{-2gy}}dx \quad \text{because} \quad \tfrac{1}{2}mv^2=-mgy$$

$$f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right)=\sqrt{\frac{1+\left(\frac{dy}{dx}\right)^2}{-y}}$$

$$\frac{d}{dx}\left(f-\frac{\partial f}{\partial(dy/dx)}\frac{dy}{dx}\right)=0$$

$$\frac{d}{dx}\left(\frac{1}{\sqrt{-y\left(1+\left(\frac{dy}{dx}\right)^2\right)}}\right)=0$$

Note that for the original form of Euler-Lagrange equation:

$$\left(\frac{\partial f}{\partial y}\right)_{x,\frac{dy}{dx}}-\frac{d}{dx}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\right]=0,$$

differential equation is more complicated:

$$-\frac{1}{2}\sqrt{\frac{1+\left(\frac{dy}{dx}\right)^2}{-y^3}}-\frac{d}{dx}\left(\frac{\frac{dy}{dx}}{\sqrt{-y\left(1+\left(\frac{dy}{dx}\right)^2\right)}}\right)=0$$

$$f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right)=\sqrt{\frac{1+\left(\frac{dy}{dx}\right)^2}{-y}}$$

$$\frac{d}{dx}\left(f-\frac{\partial f}{\partial(dy/dx)}\frac{dy}{dx}\right)=\left(\frac{\partial f}{\partial x}\right)$$

$$\Rightarrow\frac{d}{dx}\left(\frac{1}{\sqrt{-y\left(1+\left(\frac{dy}{dx}\right)^2\right)}}\right)=0 \qquad -y\left(1+\left(\frac{dy}{dx}\right)^2\right)=K\equiv 2a$$

$$-y\left(1+\left(\frac{dy}{dx}\right)^2\right)=K\equiv 2a$$

$$\frac{dy}{dx}=-\sqrt{\frac{2a}{-y}-1}$$

$$-\frac{dy}{\sqrt{\frac{2a}{-y}-1}}=dx$$

Let $y=-2a\sin^2\frac{\theta}{2}=a(\cos\theta-1)$

$$-\frac{dy}{\sqrt{\frac{2a}{-y}-1}}=\frac{2a\sin\frac{\theta}{2}\cos\frac{\theta}{2}\,d\theta}{\sqrt{\frac{2a}{2a\sin^2\frac{\theta}{2}}-1}}=dx$$

$$x=\int_0^\theta a(1-\cos\theta')d\theta'=a(\theta-\sin\theta)$$

Parametric equations for Brachistochrone:

$$x=a(\theta-\sin\theta)$$
$$y=a(\cos\theta-1)$$

Parametric plot --
plot([theta-sin(theta), cos(theta)-1, theta = 0 .. Pi