PHY 711 Classical Mechanics and Mathematical Methods
11-11:50 AM MWF Olin 107

Plan for Lecture 8:

Continue reading Chapter 3 & 6

1. **Summary & review**
2. **Lagrange's equations with constraints**

PHY 711 Classical Mechanics and Mathematical Methods

MWF 11 AM-11:50 AM | OPL 107 | http://www.wfu.edu/~natalie/f16phy711/

Instructor: Natalie Holzwarth | Phone:758-5510 | Office:300 OPL | e-mail:natalie@wfu.edu

Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

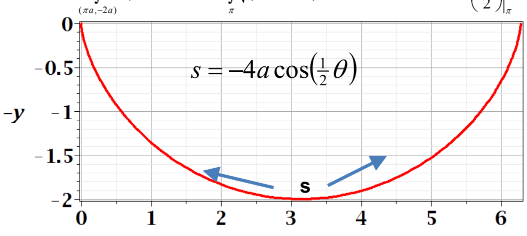
	Date	F&W Reading	Topic	Assignment	Due
1	Wed, 8/31/2016	Chap. 1	Review of basic principles	#1	9/7/2016
2	Fri, 9/02/2016	Chap. 1	Scattering theory	#2	9/7/2016
	Mon, 9/05/2016		Labor day -- no class		
3	Wed, 9/07/2016	Chap. 1	Scattering theory	#3	9/9/2016
4	Fri, 9/09/2016	Chap. 1 & 2	Scattering theory and rotations	#4	9/12/2016
5	Mon, 9/12/2016	Chap. 3	Calculus of variations	#5	9/14/2016
6	Wed, 9/14/2016	Chap. 3	Calculus of variations	#6	9/16/2016
7	Fri, 9/16/2016	Chap. 3	Lagrangian mechanics	#7	9/19/2016
8	Mon, 9/19/2016	Chap. 3 and 6	Lagrangian mechanics and constraints	#8	9/21/2016
9	Wed, 9/21/2016				

Summary of Lagrangian mechanics

$$L = T - U = L\left(\{q_\sigma(t)\},\{\dot{q}_\sigma(t)\},t\right) \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

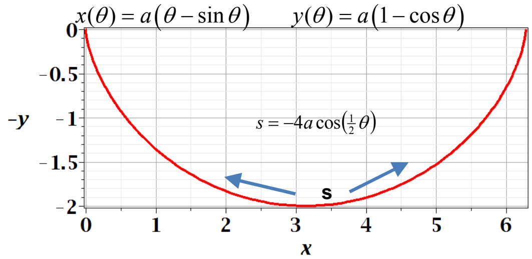
Consider a generalized coordinate formed from an extended Brachistochrone trajectory: $x(\theta) = a(\theta - \sin\theta)$ and $y(\theta) = a(1-\cos\theta)$

$$s = \int_{(\pi a,-2a)}^{(x(\theta),y(\theta))} \sqrt{dx'^2 + dy'^2} = a\int_{\pi}^{\theta}\sqrt{(1-\cos\theta')^2 + \sin^2\theta'}\,d\theta' = -4a\cos\left(\frac{\theta'}{2}\right)\Big|_{\pi}^{\theta}$$



Comment on problem Lagrangian formulation of Brachistochrone motion:

$$x(\theta) = a(\theta - \sin\theta) \qquad y(\theta) = a(1 - \cos\theta)$$



Lagrangian for mass frictionlessly traveling along s:

$$L(s(t), \dot{s}(t)) = \tfrac{1}{2}m\dot{s}^2 - mgy = \tfrac{1}{2}m\dot{s}^2 - mg2a\left(1 - \left(\frac{s}{4a}\right)^2\right)$$

Lagrangian for mass traveling along s:

$$L(s(t), \dot{s}(t)) = \tfrac{1}{2}m\dot{s}^2 - mgy = \tfrac{1}{2}m\dot{s}^2 - mg2a\left(1 - \left(\frac{s}{4a}\right)^2\right)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{s}} - \frac{\partial L}{\partial s} = 0$$

$$\Rightarrow m\ddot{s} = -\frac{mg}{4a}s$$

$$\Rightarrow \ddot{s} = -\frac{g}{4a}s$$

$$s(t) = s_0 + A\sin\left(\sqrt{\frac{g}{4a}}t\right)$$

Comments on generalized coordinates:

$$L = L(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

Here we have assumed that the generalized coordinates q_σ are independent. Now consider the possibility that the coordinates are related through constraint equations of the form:

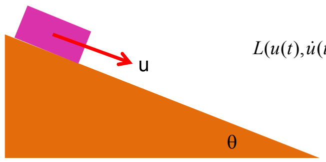
Lagrangian : $L = L(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t)$

Constraints : $f_j = f_j(\{q_\sigma(t)\}, t) = 0$

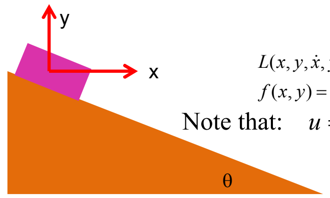
Lagrange multipliers

Modified Euler - Lagrange equations : $\displaystyle \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} + \sum_j \lambda_j \frac{\partial f_j}{\partial q_\sigma} = 0$

Simple example:



$$L(u(t),\dot{u}(t)) = \tfrac{1}{2}m\dot{u}^2 + mgu\sin\theta$$



$$L(x,y,\dot{x},\dot{y}) = \tfrac{1}{2}m\left(\dot{x}^2+\dot{y}^2\right) - mgy$$
$$f(x,y) = \sin\theta\, x + \cos\theta\, y = 0$$

Note that: $u = x\cos\theta - y\sin\theta$

Case 1:

$$L(u(t),\dot{u}(t)) = \tfrac{1}{2}m\dot{u}^2 + mgu\sin\theta$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{u}} - \frac{\partial L}{\partial u} = 0 = m\ddot{u} - mg\sin\theta = 0 \qquad \Rightarrow \ddot{u} = g\sin\theta$$

Case 2:

$$L(x,y,\dot{x},\dot{y}) = \tfrac{1}{2}m\left(\dot{x}^2+\dot{y}^2\right) - mgy$$

$$f(x,y) = \sin\theta\, x + \cos\theta\, y = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} + \lambda\frac{\partial f}{\partial x} = 0 = m\ddot{x} + \lambda\sin\theta$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} + \lambda\frac{\partial f}{\partial y} = 0 = m\ddot{y} + mg + \lambda\cos\theta$$

$$\sin\theta\,\ddot{x} + \cos\theta\,\ddot{y} = 0$$

$$\Rightarrow \lambda = mg\cos\theta$$

Force of constraint; normal to incline

$$\left(\cos\theta\,\ddot{x} - \sin\theta\,\ddot{y}\right) = g\sin\theta$$

Rational for Lagrange multipliers

Recall Hamilton's principle:

$$S = \int_{t_i}^{t_f} L\left(\{q_\sigma(t)\},\{\dot{q}_\sigma(t)\},t\right)dt$$

$$\delta S = 0 = \int_{t_i}^{t_f}\left(\sum_\sigma\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma}\right)\delta q_\sigma\right)dt$$

With constraints: $f_j = f_j\left(\{q_\sigma(t)\},t\right) = 0$

Variations δq_σ are no longer independent.

$$\delta f_j = 0 = \sum_\sigma \frac{\partial f_j}{\partial q_\sigma}\delta q_\sigma \quad \text{at each } t$$

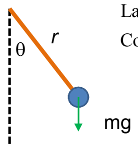
$\Rightarrow$ Add 0 to Euler-Lagrange equations in the form:

$$\sum_j \lambda_j \sum_\sigma \frac{\partial f_j}{\partial q_\sigma}\delta q_\sigma$$

Euler-Lagrange equations with constraints:

$$\text{Lagrangian}: \quad L = L(\{q_\sigma(t)\},\{\dot{q}_\sigma(t)\},t)$$
$$\text{Constraints}: \quad f_j = f_j(\{q_\sigma(t)\},t) = 0$$
$$\text{Modified Euler-Lagrange equations}: \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} + \sum_j \lambda_j \frac{\partial f_j}{\partial q_\sigma} = 0$$

Example:



$$\text{Lagrangian}: \quad L = \tfrac{1}{2} m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + mgr\cos\theta$$
$$\text{Constraints}: \quad f = r - \ell = 0$$

Example continued:

$$\text{Lagrangian}: \quad L = \tfrac{1}{2} m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + mgr\cos\theta$$
$$\text{Constraints}: \quad f = r - \ell = 0$$

$$\frac{d}{dt} m\dot{r} - mr\dot{\theta}^2 - mg\cos\theta + \lambda = 0$$
$$\frac{d}{dt} mr^2\dot{\theta} + mgr\sin\theta = 0$$
$$\dot{r} = 0 = \ddot{r} \qquad r = \ell$$
$$\Rightarrow \ddot{\theta} = -\frac{g}{\ell}\sin\theta$$
$$\Rightarrow \lambda = m\ell\dot{\theta}^2 + mg\cos\theta$$

Another example:

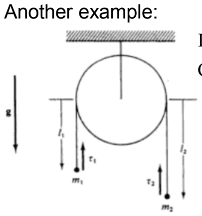
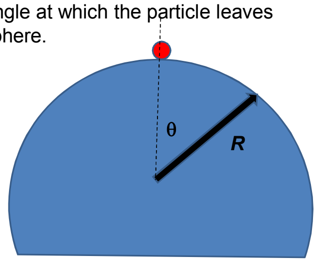


Figure 19.1 Atwood's machine.

$$\text{Lagrangian}: \quad L = \tfrac{1}{2} m_1\dot{\ell}_1^2 + \tfrac{1}{2} m_2\dot{\ell}_2^2 + m_1 g\ell_1 + m_2 g\ell_2$$
$$\text{Constraints}: \quad f = \ell_1 + \ell_2 - \ell = 0$$
$$\frac{d}{dt} m_1\dot{\ell}_1 - m_1 g + \lambda = 0$$
$$\frac{d}{dt} m_2\dot{\ell}_2 - m_2 g + \lambda = 0$$
$$\dot{\ell}_1 + \dot{\ell}_2 = 0 = \ddot{\ell}_1 + \ddot{\ell}_2$$
$$\Rightarrow \lambda = \frac{2m_1 m_2}{m_1 + m_2} g$$
$$\ddot{\ell}_1 = -\ddot{\ell}_2 = \frac{m_1 - m_2}{m_1 + m_2} g$$

Another example:

A particle of mass *m* starts at rest on top of a smooth fixed hemisphere of radius *R*. Find the angle at which the particle leaves the hemisphere.



Example continued

Constraint Equation: $f(r,\theta) = r - R = 0$

Lagrangian: $L(r,\theta,\dot{r},\dot{\theta}) = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + mgr\left(1-\cos\theta\right)$

Euler - Lagrangian equations :

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} + \lambda\frac{\partial f}{\partial r} = 0 \qquad mr\dot{\theta}^2 + mg\left(1-\cos\theta\right) - m\ddot{r} + \lambda = 0$$

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} + \lambda\frac{\partial f}{\partial \theta} = 0 \qquad mgr\sin\theta - mr^2\ddot{\theta} - 2mr\dot{r}\dot{\theta} = 0$$

Example continued

$$mr\dot{\theta}^2 + mg\left(1-\cos\theta\right) - m\ddot{r} + \lambda = 0$$

$$mgr\sin\theta - mr^2\ddot{\theta} - 2mr\dot{r}\dot{\theta} = 0$$

Also note, from conservation of energy:

$$\frac{1}{2}mR^2\dot{\theta}^2 - mgR\left(1-\cos\theta\right) = 0$$

$$\dot{\theta}^2 = \frac{2g}{R}\left(1-\cos\theta\right)$$

Using constraint:

$$mR\dot{\theta}^2 + mg\left(1-\cos\theta\right) + \lambda = 0$$

$$mgR\sin\theta - mR^2\ddot{\theta} = 0$$

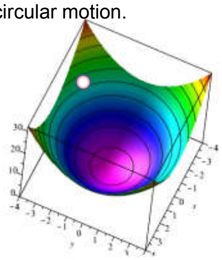
$$\ddot{\theta} = \frac{g}{R}\sin\theta \qquad \lambda = mg\cos\theta - mR\dot{\theta}^2 - mg$$

In this case λ, is not exactly the force of constraint F_C

$$F_C = mg\ \cos\theta - mR\dot{\theta}^2 = mg\left(3\cos\theta - 2\right) \quad \text{for } F_C \ge 0$$

$$\Rightarrow \text{for } \cos\theta = \frac{2}{3} \quad \text{or } \theta = 48.2 \text{ deg, object falls from hemisphere}$$

Consider a particle of mass *m* moving frictionlessly on a parabola *z=c(x²+y²)* under the influence of gravity. Find the equations of motion, particularly showing stable circular motion.



$$L(x,y,\dot{x},\dot{y})=\frac{m}{2}\left(\dot{x}^2+\dot{y}^2+4c^2\left(x\dot{x}+y\dot{y}\right)^2\right)-mgc\left(x^2+y^2\right)$$

$$L(x,y,\dot{x},\dot{y})=\frac{m}{2}\left(\dot{x}^2+\dot{y}^2+4c^2\left(x\dot{x}+y\dot{y}\right)^2\right)-mgc\left(x^2+y^2\right)$$

Transform to polar coordinates;

$$x=r\cos\phi \qquad y=r\sin\phi$$

$$L(r,\phi,\dot{r},\dot{\phi})=\frac{m}{2}\left(\dot{r}^2+r^2\dot{\phi}^2+4c^2r^2\dot{r}^2\right)-mgcr^2$$

Euler-Lagrange equations

$$\frac{\partial L}{\partial\phi}-\frac{d}{dt}\frac{\partial L}{\partial\dot{\phi}}=0 \quad\Rightarrow\quad 0-\frac{d}{dt}mr^2\dot{\phi}=0$$

$$\frac{\partial L}{\partial r}-\frac{d}{dt}\frac{\partial L}{\partial\dot{r}}=0 \qquad\Rightarrow\quad \text{Let } mr^2\dot{\phi}\equiv\ell_z \quad \text{(constant)}$$

$$L(r,\phi,\dot{r},\dot{\phi})=\frac{m}{2}\left(\dot{r}^2+r^2\dot{\phi}^2+4c^2r^2\dot{r}^2\right)-mgcr^2$$

$$\frac{\partial L}{\partial r}-\frac{d}{dt}\frac{\partial L}{\partial\dot{r}}=0$$

$$mr\dot{\phi}^2+4m\dot{r}^2c^2r-2mgcr-\frac{d}{dt}\left(m\dot{r}\left(1+4c^2r^2\right)\right)=0$$

$$\underbrace{\frac{\ell_z^{\,2}}{mr^3}-2mgcr}-\frac{d}{dt}\left(m\dot{r}\left(1+2c^2r^2\right)\right)=0$$

Stable solution when these terms add to 0:

$$\ell_z=mr_0^2\sqrt{2gc}=mz_0\sqrt{\frac{2g}{c}}$$

Analysis of stable (circular) motion

$$\frac{\ell_z^{\,2}}{mr^3} - 2mgcr - \frac{d}{dt}\left(m\dot{r}\left(1+2c^2r^2\right)\right) = 0$$

$$\ell_z = mr_0^2\sqrt{2gc} = mz_0\sqrt{\frac{2g}{c}}$$

Let $r = r_0 + \delta r$ keeping terms to linear order:

$$-8mgc\delta r - m\delta\ddot{r}\left(1+2c^2r_0^2\right) = 0$$

$$\delta\ddot{r} = -\frac{8gc}{1+2c^2r_0^2}\delta r$$

$$\Rightarrow \delta r = A\cos\left(\sqrt{\frac{8gc}{1+2c^2r_0^2}}t + \alpha\right)$$