PHY 711 Classical Mechanics and Mathematical Methods
9-9:50 AM MWF Olin 107
Plan for Lecture 10:

Continue reading Chapter 3 & 6

1. **Constants of the motion**
2. **Conserved quantities**
3. **Legendre transformations**

Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W Reading	Topic	Assignment	Due
1	Mon, 8/28/2017	Chap. 1	Introduction	#1	9/6/2017
2	Wed, 8/30/2017	Chap. 1	Scattering theory	#2	9/6/2017
3	Fri, 9/01/2017	Chap. 1	Scattering theory		
4	Mon, 9/04/2017	Chap. 1	Scattering theory	#3	9/6/2017
5	Wed, 9/06/2017	Chap. 2	Physics in an non-inertial reference frame	#4	9/8/2017
6	Fri, 9/08/2017	Chap. 3	Calculus of variations	#5	9/11/2017
7	Mon, 9/11/2017	Chap.3	Calculus of variations	#6	9/13/2017
8	Wed, 9/13/2017	Chap. 3	Lagrangian Mechanics	#7	9/15/2017
9	Fri, 9/15/2017	Chap. 3 and 6	Lagrangian mechanics and constraints	#8	9/20/2017
10	Mon, 9/18/2017	Chap. 3 and 6	Constants of the motion		
11	Wed, 9/20/2017				
12	Fri, 9/22/2017				
13	Mon, 9/25/2017				
14	Wed, 9/27/2017				
15	Fri, 9/29/2017				
	Mon, 10/02/2017		*Take-home exam -- No class*		

Summary of Lagrangian formalism (without constraints)

For independent generalized coordinates $q_\sigma(t)$:

$$L = L(\{q_\sigma(t)\},\{\dot{q}_\sigma(t)\},t)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

Note that if $\frac{\partial L}{\partial q_\sigma} = 0$, then $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} = 0$

$$\Rightarrow \frac{\partial L}{\partial \dot{q}_\sigma} = \text{(constant)}$$

Examples of constants of the motion:

Example 1: one-dimensional potential:

$$L = \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - V(z)$$

$$\Rightarrow \frac{d}{dt}m\dot{x} = 0 \qquad \Rightarrow m\dot{x} \equiv p_x \text{ (constant)}$$

$$\Rightarrow \frac{d}{dt}m\dot{y} = 0 \qquad \Rightarrow m\dot{y} \equiv p_y \text{ (constant)}$$

$$\Rightarrow \frac{d}{dt}m\dot{z} = -\frac{\partial V}{\partial z}$$

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Examples of constants of the motion:

Example 2: Motion in a central potential

$$L = \tfrac{1}{2}m\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) - V(r)$$

$$\Rightarrow \frac{d}{dt}mr^2\dot{\varphi} = 0 \qquad \Rightarrow mr^2\dot{\varphi} \equiv p_\varphi \text{ (constant)}$$

$$\Rightarrow \frac{d}{dt}m\dot{r} = mr\dot{\varphi}^2 - \frac{\partial V}{\partial r} = \frac{p_\varphi{}^2}{mr^3} - \frac{\partial V}{\partial r}$$

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Recall alternative form of Euler-Lagrange equations:

Starting from:

$$L = L(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

Also note that: $\frac{dL}{dt} = \sum_\sigma \frac{\partial L}{\partial q_\sigma}\dot{q}_\sigma + \sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma}\ddot{q}_\sigma + \frac{\partial L}{\partial t}$

$$= \frac{d}{dt}\left(\sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma}\dot{q}_\sigma\right) + \frac{\partial L}{\partial t}$$

$$\Rightarrow \frac{d}{dt}\left(L - \sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma}\dot{q}_\sigma\right) = \frac{\partial L}{\partial t}$$

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Additional constant of the motion:

If $\frac{\partial L}{\partial t} = 0;$

then : $\frac{d}{dt}\left(L - \sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma}\dot{q}_\sigma \right) = \frac{\partial L}{\partial t} = 0$

$$\Rightarrow L - \sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma}\dot{q}_\sigma = -E \quad \text{(constant)}$$

Example 1 : one-dimensional potential :

$$L = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - V(z)$$

$$\Rightarrow \frac{d}{dt}\left(\tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - V(z) - m\dot{x}^2 - m\dot{y}^2 - m\dot{z}^2\right) = 0$$

$$\Rightarrow -\left(\tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + V(z)\right) = -E \quad \text{(constant)}$$

For this case, we also have $m\dot{x} \equiv p_x$ and $m\dot{y} \equiv p_y$

$$\Rightarrow E = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \tfrac{1}{2} m\dot{z}^2 + V(z)$$

Additional constant of the motion -- continued:

If $\frac{\partial L}{\partial t} = 0;$

then : $\frac{d}{dt}\left(L - \sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma}\dot{q}_\sigma \right) = \frac{\partial L}{\partial t} = 0$

$$\Rightarrow L - \sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma}\dot{q}_\sigma = -E \quad \text{(constant)}$$

Example 2 : Motion in a central potential

$$L = \tfrac{1}{2} m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - V(r)$$

$$\Rightarrow \frac{d}{dt}\left(\tfrac{1}{2} m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - V(r) - m\dot{r}^2 - mr^2\dot{\phi}^2\right) = 0$$

$$\Rightarrow -\left(\tfrac{1}{2} m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) + V(r)\right) = -E \quad \text{(constant)}$$

For this case, we also have $mr^2\dot{\varphi} \equiv p_\varphi$

$$\Rightarrow E = \frac{p_\varphi^2}{2mr^2} + \tfrac{1}{2} m\dot{r}^2 + V(r)$$

Other examples

$$L = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + \frac{q}{2c} B_0\left(-\dot{x}y + \dot{y}x\right)$$

$$\frac{\partial L}{\partial z} = 0 \qquad \Rightarrow m\dot{z} = p_z \quad \text{(constant)}$$

$$E = \sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma}\dot{q}_\sigma - L$$

$$= m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + \frac{q}{2c} B_0\left(-\dot{x}y + \dot{y}x\right)$$

$$-\tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - \frac{q}{2c} B_0\left(-\dot{x}y + \dot{y}x\right)$$

$$= \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2\right) + \frac{p_z^2}{2m}$$

Other examples

$$L = \tfrac{1}{2}m\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right) - \frac{q}{c}B_0\dot{x}y$$

$$\frac{\partial L}{\partial z} = 0 \qquad \Rightarrow m\dot{z} = p_z \quad \text{(constant)}$$

$$\frac{\partial L}{\partial x} = 0 \qquad \Rightarrow m\dot{x} = p_x \quad \text{(constant)}$$

$$E = \sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma}\dot{q}_\sigma - L$$

$$= m\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right) - \frac{q}{c}B_0\dot{x}y$$

$$-\tfrac{1}{2}m\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right) + \frac{q}{c}B_0\dot{x}y$$

$$= \tfrac{1}{2}m\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right) = \tfrac{1}{2}m\dot{y}^2 + \frac{p_x^2}{2m} + \frac{p_z^2}{2m}$$

Lagrangian picture

For independent generalized coordinates $q_\sigma(t)$:

$$L = L(\{q_\sigma(t)\},\{\dot{q}_\sigma(t)\},t)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

$\Rightarrow$ Second order differential equations for $q_\sigma(t)$

Switching variables – Legendre transformation

Mathematical transformations for continuous functions of several variables & Legendre transforms:

Simple change of variables:

$$z(x,y) \quad \Leftrightarrow \quad x(y,z)???$$

$$z(x,y) \Rightarrow dz = \left(\frac{\partial z}{\partial x}\right)_y dx + \left(\frac{\partial z}{\partial y}\right)_x dy$$

$$x(y,z) \Rightarrow dx = \left(\frac{\partial x}{\partial y}\right)_z dy + \left(\frac{\partial x}{\partial z}\right)_y dz$$

$$\text{But:} \quad \left(\frac{\partial x}{\partial y}\right)_z = -\frac{(\partial z/\partial y)_x}{(\partial z/\partial x)_y}$$

Simple change of variables -- continued:

$$z(x,y) \Rightarrow dz = \left(\frac{\partial z}{\partial x}\right)_y dx + \left(\frac{\partial z}{\partial y}\right)_x dy$$

$$x(y,z) \Rightarrow dx = \left(\frac{\partial x}{\partial y}\right)_z dy + \left(\frac{\partial x}{\partial z}\right)_y dz$$

$$\Rightarrow \left(\frac{\partial x}{\partial y}\right)_z = -\frac{(\partial z/\partial y)_x}{(\partial z/\partial x)_y} \qquad \Rightarrow \left(\frac{\partial x}{\partial z}\right)_y = \frac{1}{(\partial z/\partial x)_y}$$

Simple change of variables -- continued:

Example:

$z(x,y) = e^{x^2+y}$

$x(y,z) = (\ln z - y)^{1/2}$

$$z(x,y) \Rightarrow dz = \left(\frac{\partial z}{\partial x}\right)_y dx + \left(\frac{\partial z}{\partial y}\right)_x dy$$

$$x(y,z) \Rightarrow dx = \left(\frac{\partial x}{\partial y}\right)_z dy + \left(\frac{\partial x}{\partial z}\right)_y dz$$

$$\left(\frac{\partial x}{\partial y}\right)_z \overset{?}{=} -\frac{(\partial z/\partial y)_x}{(\partial z/\partial x)_y} \qquad \left(\frac{\partial x}{\partial z}\right)_y \overset{?}{=} \frac{1}{(\partial z/\partial x)_y}$$

$$-\frac{1}{2(\ln z - y)^{1/2}} \overset{\checkmark}{=} -\frac{e^{x^2+y}}{2xe^{x^2+y}} \qquad \frac{1}{2z(\ln z - y)^{1/2}} \overset{\checkmark}{=} \frac{1}{2xe^{x^2+y}}$$

Mathematical transformations for continuous functions of several variables & Legendre transforms continued:

$$z(x,y) \Rightarrow dz = \left(\frac{\partial z}{\partial x}\right)_y dx + \left(\frac{\partial z}{\partial y}\right)_x dy$$

Let $u \equiv \left(\frac{\partial z}{\partial x}\right)_y$ and $v \equiv \left(\frac{\partial z}{\partial y}\right)_x$

Define new function

$$w(u,y) \Rightarrow dw = \left(\frac{\partial w}{\partial u}\right)_y du + \left(\frac{\partial w}{\partial y}\right)_u dy$$

For $w = z - ux$, $dw = dz - udx - xdu = udx + vdy - udx - xdu$

$dw = -xdu + vdy$

$$\Rightarrow \left(\frac{\partial w}{\partial u}\right)_y = -x \qquad \left(\frac{\partial w}{\partial y}\right)_u = \left(\frac{\partial z}{\partial y}\right)_x = v$$

For thermodynamic functions:

Internal energy: $U = U(S,V)$

$$dU = TdS - PdV$$

$$dU = \left(\frac{\partial U}{\partial S}\right)_V dS + \left(\frac{\partial U}{\partial V}\right)_S dV$$

$$\Rightarrow T = \left(\frac{\partial U}{\partial S}\right)_V \qquad P = -\left(\frac{\partial U}{\partial V}\right)_S$$

Enthalpy: $H = H(S,P) = U + PV$

$$dH = dU + PdV + VdP = TdS + VdP = \left(\frac{\partial H}{\partial S}\right)_P dS + \left(\frac{\partial H}{\partial P}\right)_S dP$$

$$\Rightarrow T = \left(\frac{\partial H}{\partial S}\right)_P \qquad V = \left(\frac{\partial H}{\partial P}\right)_S$$

Name	Potential	Differential Form
Internal energy	$E(S,V,N)$	$dE = TdS - PdV + \mu dN$
Entropy	$S(E,V,N)$	$dS = \frac{1}{T}dE + \frac{P}{T}dV - \frac{\mu}{T}dN$
Enthalpy	$H(S,P,N) = E + PV$	$dH = TdS + VdP + \mu dN$
Helmholtz free energy	$F(T,V,N) = E - TS$	$dF = -SdT - PdV + \mu dN$
Gibbs free energy	$G(T,P,N) = F + PV$	$dG = -SdT + VdP + \mu dN$
Landau potential	$\Omega(T,V,\mu) = F - \mu N$	$d\Omega = -SdT - PdV - Nd\mu$

Lagrangian picture

For independent generalized coordinates $q_\sigma(t)$:

$$L = L(\{q_\sigma(t)\},\{\dot{q}_\sigma(t)\},t)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

$\Rightarrow$ Second order differential equations for $q_\sigma(t)$

Switching variables – Legendre transformation

Define: $H = H(\{q_\sigma(t)\},\{p_\sigma(t)\},t)$

$$H = \sum_\sigma \dot{q}_\sigma p_\sigma - L \qquad \text{where } p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma}$$

$$dH = \sum_\sigma \left(\dot{q}_\sigma dp_\sigma + p_\sigma d\dot{q}_\sigma - \frac{\partial L}{\partial q_\sigma} dq_\sigma - \frac{\partial L}{\partial \dot{q}_\sigma} d\dot{q}_\sigma\right) - \frac{\partial L}{\partial t} dt$$

Hamiltonian picture – continued

$$H = H(\{q_\sigma(t)\},\{p_\sigma(t)\},t)$$

$$H = \sum_\sigma \dot{q}_\sigma p_\sigma - L \qquad \text{where} \quad p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma}$$

$$dH = \sum_\sigma \left(\dot{q}_\sigma dp_\sigma + p_\sigma d\dot{q}_\sigma - \frac{\partial L}{\partial q_\sigma} dq_\sigma - \frac{\partial L}{\partial \dot{q}_\sigma} d\dot{q}_\sigma \right) - \frac{\partial L}{\partial t} dt$$

$$= \sum_\sigma \left(\frac{\partial H}{\partial q_\sigma} dq_\sigma + \frac{\partial H}{\partial p_\sigma} dp_\sigma \right) + \frac{\partial H}{\partial t} dt$$

$$\Rightarrow \dot{q}_\sigma = \frac{\partial H}{\partial p_\sigma} \qquad \frac{\partial L}{\partial q_\sigma} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} \equiv \dot{p}_\sigma = -\frac{\partial H}{\partial q_\sigma} \qquad \frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t}$$