PHY 711 Classical Mechanics and Mathematical Methods
9-9:50 AM MWF Olin 107

Plan for Lecture 13:

Finish reading Chapter 6

1. **Virial theorem**
2. **Canonical transformations**
3. **Hamilton-Jacobi formalism**

Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W Reading	Topic	Assignment	Due
1	Mon, 8/28/2017	Chap. 1	Introduction	#1	9/6/2017
2	Wed, 8/30/2017	Chap. 1	Scattering theory	#2	9/6/2017
3	Fri, 9/01/2017	Chap. 1	Scattering theory		
4	Mon, 9/04/2017	Chap. 1	Scattering theory	#3	9/6/2017
5	Wed, 9/06/2017	Chap. 2	Physics in an non-inertial reference frame	#4	9/8/2017
6	Fri, 9/08/2017	Chap. 3	Calculus of variations	#5	9/11/2017
7	Mon, 9/11/2017	Chap.3	Calculus of variations	#6	9/13/2017
8	Wed, 9/13/2017	Chap. 3	Lagrangian Mechanics	#7	9/15/2017
9	Fri, 9/15/2017	Chap. 3 and 6	Lagrangian mechanics and constraints	#8	9/20/2017
10	Mon, 9/18/2017	Chap. 3 and 6	Constants of the motion		
11	Wed, 9/20/2017	Chap. 3 and 6	Hamiltonian formalism	#9	9/27/2017
12	Fri, 9/22/2017	Chap. 3 and 6	Liouville equation		
13	Mon, 9/25/2017	Chap. 6	Canonical transformations		
14	Wed, 9/27/2017	Chap. 4	Motion from Small oscillations about equilibrium		
15	Fri, 9/29/2017	Chap. 1-4,6	Review		
	Mon, 10/02/2017		*Take-home exam -- No class*		
	Wed, 10/04/2017		*Take-home exam -- No class*		
16	Fri, 10/06/2017				

Virial theorem (Clausius ~ 1860)

$$2\langle T\rangle = -\left\langle \sum_\sigma \mathbf{F}_\sigma \cdot \mathbf{r}_\sigma \right\rangle$$

Proof:

Define: $A \equiv \sum_\sigma \mathbf{p}_\sigma \cdot \mathbf{r}_\sigma$

$$\frac{dA}{dt} = \sum_\sigma \left(\dot{\mathbf{p}}_\sigma \cdot \mathbf{r}_\sigma + \mathbf{p}_\sigma \cdot \dot{\mathbf{r}}_\sigma\right) = \sum_\sigma \mathbf{F}_\sigma \cdot \mathbf{r}_\sigma + 2T$$

Because $\dot{\mathbf{p}}_\sigma = \mathbf{F}_\sigma$

$$\left\langle \frac{dA}{dt} \right\rangle = \left\langle \sum_\sigma \mathbf{F}_\sigma \cdot \mathbf{r} \right\rangle + 2\langle T\rangle$$

$$\left\langle \frac{dA}{dt} \right\rangle = \frac{1}{\tau}\int_0^\tau \frac{dA(t)}{dt}dt = \frac{A(\tau)-A(0)}{\tau} \Rightarrow 0$$

Note that this implies that the motion is bounded

$$\Rightarrow \left\langle \sum_\sigma \mathbf{F}_\sigma \cdot \mathbf{r} \right\rangle + 2\langle T\rangle = 0$$

Examples of the Virial Theorem $2\langle T\rangle = -\left\langle \sum_\sigma \mathbf{F}_\sigma \cdot \mathbf{r}_\sigma \right\rangle$

Harmonic oscillator:

$\mathbf{F} = -kx\hat{\mathbf{x}} \qquad T = \frac{1}{2}m\dot{x}^2 \qquad \langle m\dot{x}^2\rangle = \langle kx^2\rangle$

Check: for $x(t) = A\sin\left(\sqrt{\frac{k}{m}}t + \alpha\right)$

$$\langle m\dot{x}^2\rangle = \frac{1}{2}kA^2 = \langle kx^2\rangle$$

Examples of the Virial Theorem $2\langle T\rangle = -\left\langle \sum_\sigma \mathbf{F}_\sigma \cdot \mathbf{r}_\sigma \right\rangle$

Circular orbit due to gravitational field of massive object:

$\mathbf{F} = -\frac{GMm}{r^2}\hat{\mathbf{r}} \qquad T = \frac{1}{2}mr^2\omega^2 \qquad \langle mr^2\omega^2\rangle = \left\langle \frac{GMm}{r}\right\rangle$

Check: for $r\omega^2 = \frac{GM}{r^2} \qquad \Rightarrow \langle mr^2\omega^2\rangle = \left\langle \frac{GMm}{r}\right\rangle$

Hamiltonian formalism and the canonical equations of motion:

$$H = H(\{q_\sigma(t)\},\{p_\sigma(t)\},t)$$

Canonical equations of motion

$$\frac{dq_\sigma}{dt} = \frac{\partial H}{\partial p_\sigma}$$

$$\frac{dp_\sigma}{dt} = -\frac{\partial H}{\partial q_\sigma}$$

Notion of "Canonical" transformations

$$q_\sigma = q_\sigma\left(\{Q_1 \cdots Q_n\},\{P_1 \cdots P_n\},t\right) \qquad \text{for each } \sigma$$
$$p_\sigma = p_\sigma\left(\{Q_1 \cdots Q_n\},\{P_1 \cdots P_n\},t\right) \qquad \text{for each } \sigma$$

For some $\tilde{H}$ and F, using Legendre transformations

$$\sum_\sigma p_\sigma \dot{q}_\sigma - H\left(\{q_\sigma\},\{p_\sigma\},t\right) = \sum_\sigma P_\sigma \dot{Q}_\sigma - \tilde{H}\left(\{Q_\sigma\},\{P_\sigma\},t\right) + \frac{d}{dt}F\left(\{q_\sigma\},\{Q_\sigma\},t\right)$$

Apply Hamilton's principle:

$$\delta \int_{t_i}^{t_f} \left[\sum_\sigma P_\sigma \dot{Q}_\sigma - \tilde{H}\left(\{Q_\sigma\},\{P_\sigma\},t\right) + \frac{d}{dt}F\left(\{q_\sigma\},\{Q_\sigma\},t\right) \right] dt = 0$$

$$\delta \int_{t_i}^{t_f} \left[\frac{d}{dt}F\left(\{q_\sigma\},\{Q_\sigma\},t\right) \right] dt = \int_{t_i}^{t_f} \left[\frac{d}{dt}\delta F\left(\{q_\sigma\},\{Q_\sigma\},t\right) \right] dt$$

$$= \delta F(t_f) - \delta F(t_i) = 0 \qquad \text{and} \qquad \dot{Q}_\sigma = \frac{\partial \tilde{H}}{\partial P_\sigma} \qquad \dot{P}_\sigma = -\frac{\partial \tilde{H}}{\partial Q_\sigma}$$

Some relations between old and new variables:

$$\sum_\sigma p_\sigma \dot{q}_\sigma - H(\{q_\sigma\},\{p_\sigma\},t) =$$
$$\sum_\sigma P_\sigma \dot{Q}_\sigma - \tilde{H}(\{Q_\sigma\},\{P_\sigma\},t) + \frac{d}{dt}F(\{q_\sigma\},\{Q_\sigma\},t)$$

$$\frac{d}{dt}F(\{q_\sigma\},\{Q_\sigma\},t) = \sum_\sigma \left(\left(\frac{\partial F}{\partial q_\sigma}\right)\dot{q}_\sigma + \left(\frac{\partial F}{\partial Q_\sigma}\right)\dot{Q}_\sigma \right) + \frac{\partial F}{\partial t}$$

$$\Rightarrow \sum_\sigma \left(p_\sigma - \left(\frac{\partial F}{\partial q_\sigma}\right) \right)\dot{q}_\sigma - H\left(\{q_\sigma\},\{p_\sigma\},t\right) =$$
$$\sum_\sigma \left(P_\sigma + \left(\frac{\partial F}{\partial Q_\sigma}\right) \right)\dot{Q}_\sigma - \tilde{H}\left(\{Q_\sigma\},\{P_\sigma\},t\right) + \frac{\partial F}{\partial t}$$

$$\sum_\sigma \left(p_\sigma - \left(\frac{\partial F}{\partial q_\sigma}\right) \right)\dot{q}_\sigma - H(\{q_\sigma\},\{p_\sigma\},t) =$$
$$\sum_\sigma \left(P_\sigma + \left(\frac{\partial F}{\partial Q_\sigma}\right) \right)\dot{Q}_\sigma - \tilde{H}(\{Q_\sigma\},\{P_\sigma\},t) + \frac{\partial F}{\partial t}$$

$$\Rightarrow p_\sigma = \left(\frac{\partial F}{\partial q_\sigma}\right) \qquad P_\sigma = -\left(\frac{\partial F}{\partial Q_\sigma}\right)$$

$$\Rightarrow \tilde{H}(\{Q_\sigma\},\{P_\sigma\},t) = H(\{q_\sigma\},\{p_\sigma\},t) + \frac{\partial F}{\partial t}$$

Note that it is conceivable that if we were extraordinarily clever, we could find all of the constants of the motion!

$$\dot{Q}_\sigma = \frac{\partial \tilde{H}}{\partial P_\sigma} \qquad \dot{P}_\sigma = -\frac{\partial \tilde{H}}{\partial Q_\sigma}$$

$$\text{Suppose: } \dot{Q}_\sigma = \frac{\partial \tilde{H}}{\partial P_\sigma} = 0 \text{ and } \dot{P}_\sigma = -\frac{\partial \tilde{H}}{\partial Q_\sigma} = 0$$

$$\Rightarrow Q_\sigma, P_\sigma \quad \text{are constants of the motion}$$

Possible solution – Hamilton-Jacobi theory:

$$\text{Suppose: } \quad F(\{q_\sigma\},\{Q_\sigma\},t) \Rightarrow -\sum_\sigma P_\sigma Q_\sigma + S(\{q_\sigma\},\{P_\sigma\},t)$$

$$\sum_\sigma p_\sigma \dot{q}_\sigma - H(\{q_\sigma\},\{p_\sigma\},t) =$$

$$\sum_\sigma P_\sigma \dot{Q}_\sigma - \tilde{H}(\{Q_\sigma\},\{P_\sigma\},t) + \frac{d}{dt}\left(-\sum_\sigma P_\sigma Q_\sigma + S(\{q_\sigma\},\{P_\sigma\},t)\right)$$

$$= -\tilde{H}(\{Q_\sigma\},\{P_\sigma\},t) - \sum_\sigma \dot{P}_\sigma Q_\sigma + \sum_\sigma \left(\frac{\partial S}{\partial q_\sigma}\dot{q}_\sigma + \frac{\partial S}{\partial P_\sigma}\dot{P}_\sigma\right) + \frac{\partial S}{\partial t}$$

Solution:

$$p_\sigma = \frac{\partial S}{\partial q_\sigma} \qquad Q_\sigma = \frac{\partial S}{\partial P_\sigma}$$

$$\tilde{H}(\{Q_\sigma\},\{P_\sigma\},t) = H(\{q_\sigma\},\{p_\sigma\},t) + \frac{\partial S}{\partial t}$$

When the dust clears:

Assume $\{Q_\sigma\},\{P_\sigma\},\tilde{H}$ are constants; choose $\tilde{H} = 0$

Need to find $S(\{q_\sigma\},\{P_\sigma\},t)$

$$p_\sigma = \frac{\partial S}{\partial q_\sigma} \qquad Q_\sigma = \frac{\partial S}{\partial P_\sigma}$$

$$\Rightarrow H\left(\{q_\sigma\},\left\{\frac{\partial S}{\partial q_\sigma}\right\},t\right) + \frac{\partial S}{\partial t} = 0$$

Note: S is the "action":

$$\sum_\sigma p_\sigma \dot{q}_\sigma - H(\{q_\sigma\},\{p_\sigma\},t) =$$

$$\sum_\sigma \underbrace{P_\sigma \dot{Q}_\sigma}_{0} - \underbrace{\tilde{H}(\{Q_\sigma\},\{P_\sigma\},t)}_{0} + \frac{d}{dt}\left(\underbrace{-\sum_\sigma P_\sigma Q_\sigma}_{0} + S(\{q_\sigma\},\{P_\sigma\},t)\right)$$

$$\sum_\sigma p_\sigma \dot{q}_\sigma - H(\{q_\sigma\},\{p_\sigma\},t) =$$

$$\sum_\sigma P_\sigma \dot{Q}_\sigma - \tilde{H}(\{Q_\sigma\},\{P_\sigma\},t) + \frac{d}{dt}\left(-\sum_\sigma P_\sigma Q_\sigma + S(\{q_\sigma\},\{P_\sigma\},t)\right)$$

(0   0   0)

$$\int_{t_i}^{t_f}\left(\sum_\sigma p_\sigma \dot{q}_\sigma - H(\{q_\sigma\},\{p_\sigma\},t)\right)dt = \int_{t_i}^{t_f}\left(\frac{d}{dt}\left(S(\{q_\sigma\},\{P_\sigma\},t)\right)\right)dt$$

$$= S(\{q_\sigma\},\{P_\sigma\},t)\Big|_{t_i}^{t_f}$$

Differential equation for ***S***:

$$H\left(\{q_\sigma\},\left\{\frac{\partial S}{\partial q_\sigma}\right\},t\right) + \frac{\partial S}{\partial t} = 0$$

Example: $\quad H(\{q\},\{p\},t) = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2$

Hamilton - Jacobi Eq: $\quad H\left(\{q\},\left\{\frac{\partial S}{\partial q}\right\},t\right) + \frac{\partial S}{\partial t} = 0$

$$\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + \frac{1}{2}m\omega^2 q^2 + \frac{\partial S}{\partial t} = 0$$

Assume: $\quad S(q,t) \equiv W(q) - Et \qquad (E \text{ constant})$

Continued:

$$\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + \frac{1}{2}m\omega^2 q^2 + \frac{\partial S}{\partial t} = 0$$

Assume: $\quad S(q,t) \equiv W(q) - Et \qquad (E \text{ constant})$

$$\frac{1}{2m}\left(\frac{dW}{dq}\right)^2 + \frac{1}{2}m\omega^2 q^2 = E$$

$$\frac{dW}{dq} = \sqrt{2mE - (m\omega)^2 q^2}$$

$$W(q) = \int \sqrt{2mE - (m\omega)^2 q^2}\, dq$$

Continued:

$$W(q)=\int\sqrt{2mE-(m\omega)^2q^2}\,dq$$
$$=\frac{1}{2}q\sqrt{2mE-(m\omega)^2q^2}+\frac{E}{\omega}\sin^{-1}\left(\frac{m\omega q}{\sqrt{2mE}}\right)+C$$
$$S(q,E,t)=\frac{1}{2}q\sqrt{2mE-(m\omega)^2q^2}+\frac{E}{\omega}\sin^{-1}\left(\frac{m\omega q}{\sqrt{2mE}}\right)-Et$$
$$\frac{\partial S}{\partial E}=Q=\frac{1}{\omega}\sin^{-1}\left(\frac{m\omega q}{\sqrt{2mE}}\right)-t$$
$$\Rightarrow q(t)=\frac{\sqrt{2mE}}{m\omega}\sin(\omega(t+Q))$$

Another example of Hamilton Jacobi equations

Example: $H\left(\{q\},\{p\},t\right)=\frac{p^2}{2m}+mgy$

Assume $y(0)=h;\quad p(0)=0$

Hamilton-Jacobi Eq: $H\left(\{q\},\left\{\frac{\partial S}{\partial q}\right\},t\right)+\frac{\partial S}{\partial t}=0$

$$\frac{1}{2m}\left(\frac{\partial S}{\partial y}\right)^2+mgy+\frac{\partial S}{\partial t}=0$$

Assume: $S(y,t)\equiv W(y)-Et \qquad (E\ \ \text{constant})$

Example: $H\left(\{q\},\{p\},t\right)=\frac{p^2}{2m}+mgy$

Assume $y(0)=h;\quad p(0)=0$

$$\frac{1}{2m}\left(\frac{dW}{dy}\right)^2+mgy=E\equiv mgh$$
$$W(y)=m\int_y^h\sqrt{2g(h-y')}\,dy'=\frac{2}{3}m\sqrt{2g}\,(h-y)^{3/2}$$
$$S(y,t)=W(y)-Et=\frac{2}{3}m\sqrt{2g}\,(h-y)^{3/2}-mght$$

Check action:

For this case: $y(t) = h - \frac{1}{2}gt^2$

$$S = \int_0^t \left(\frac{1}{2}m\dot{y}^2 - mgy \right) dt' = \frac{1}{3}mg^2t^3 - mght$$

$$S(y,t) = W(y) - Et = \frac{2}{3}m\sqrt{2g}\left(h-y\right)^{3/2} - mght$$

Recap --

Lagrangian picture

For independent generalized coordinates $q_\sigma(t)$:

$$L = L(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

$\Rightarrow$ Second order differential equations for $q_\sigma(t)$

Hamiltonian picture

$$H = H(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t)$$

$$\frac{dq_\sigma}{dt} = \frac{\partial H}{\partial p_\sigma} \qquad \frac{dp_\sigma}{dt} = -\frac{\partial H}{\partial q_\sigma}$$

$\Rightarrow$ Coupled first order differential equations for

$q_\sigma(t)$ and $p_\sigma(t)$