PHY 711 Classical Mechanics and Mathematical Methods
9-9:50 AM MWF Olin 107

Plan for Lecture 17:

Rotational motion (Chapter 5)

1. Rigid body motion
2. Moment of inertia tensor
3. Torque free motion

Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W Reading	Topic	Assignment	Due
1	Mon, 8/28/2017	Chap. 1	Introduction	#1	9/6/2017
2	Wed, 8/30/2017	Chap. 1	Scattering theory	#2	9/6/2017
3	Fri, 9/01/2017	Chap. 1	Scattering theory		
10	Mon, 9/18/2017	Chap. 3 and 6	Constants of the motion		
11	Wed, 9/20/2017	Chap. 3 and 6	Hamiltonian formalism	#9	9/27/2017
12	Fri, 9/22/2017	Chap. 3 and 6	Liouville equation		
13	Mon, 9/25/2017	Chap. 6	Canonical transformations		
14	Wed, 9/27/2017	Chap. 4	Motion from Small oscillations about equilibrium		
15	Fri, 9/29/2017	Chap. 1-4,6	Review		
	Mon, 10/02/2017		*Take-home exam -- No class*		
	Wed, 10/04/2017		*Take-home exam -- No class*		
16	Fri, 10/06/2017	Chap. 4	Vibrational modes		
17	Mon, 10/09/2017	Chap. 5	Rigid body motion	#10	10/16/2017
18	Wed, 10/11/2017				
	Fri, 10/13/2017		*Fall break -- No class*		
19	Mon, 10/16/2017				
20	Wed, 10/18/2017				
21	Fri, 10/20/2017				

Comparison of analysis in "inertial frame" versus "non-inertial frame"

Denote by $\hat{e}_i^0$ a fixed coordinate system

Denote by $\hat{e}_i$ a moving coordinate system

For an arbitrary vector $\mathbf{V}$: $\mathbf{V} = \sum_{i=1}^{3} V_i^0 \hat{e}_i^0 = \sum_{i=1}^{3} V_i \hat{e}_i$

$$\left(\frac{d\mathbf{V}}{dt}\right)_{inertial} = \sum_{i=1}^{3} \frac{dV_i^0}{dt}\hat{e}_i^0 = \sum_{i=1}^{3} \frac{dV_i}{dt}\hat{e}_i + \sum_{i=1}^{3} V_i \frac{d\hat{e}_i}{dt}$$

Define: $\left(\frac{d\mathbf{V}}{dt}\right)_{body} \equiv \sum_{i=1}^{3} \frac{dV_i}{dt}\hat{e}_i$

$$\Rightarrow \left(\frac{d\mathbf{V}}{dt}\right)_{inertial} = \left(\frac{d\mathbf{V}}{dt}\right)_{body} + \sum_{i=1}^{3} V_i \frac{d\hat{e}_i}{dt}$$

The physics of rigid body motion; body fixed frame vs inertial frame; results from Chapter 2:

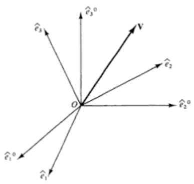


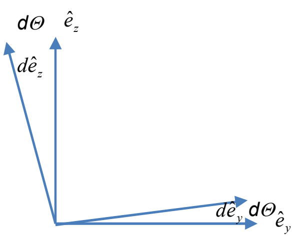
Figure 6.1 Transformation to a rotating coordinate system.

Let **V** be a general vector, e.g., the position of a particle. This vector can be characterized by its components with respect to either orthonormal triad. Thus we can write

$$\mathbf{V} = \sum_{i=1}^{3} V_i^0 \hat{e}_i^0 \qquad (6.1a)$$

$$\mathbf{V} = \sum_{i=1}^{3} V_i \hat{e}_i \qquad (6.1b)$$

Properties of the frame motion (rotation):



$$d\hat{e}_y = d\Theta \hat{e}_z$$
$$d\hat{e}_z = -d\Theta \hat{e}_y$$
$$\Rightarrow d\hat{\mathbf{e}} = d\mathbf{\Theta} \times \hat{\mathbf{e}}$$
$$\frac{d\hat{\mathbf{e}}}{dt} = \frac{d\mathbf{\Theta}}{dt} \times \hat{\mathbf{e}}$$
$$\frac{d\hat{\mathbf{e}}}{dt} = \boldsymbol{\omega} \times \hat{\mathbf{e}}$$

$$\begin{pmatrix} d\hat{e}_y \\ d\hat{e}_z \end{pmatrix} = \begin{pmatrix} \cos(d\Theta) & \sin(d\Theta) \\ -\sin(d\Theta) & \cos(d\Theta) \end{pmatrix} \begin{pmatrix} \hat{e}_y \\ \hat{e}_z \end{pmatrix} - \begin{pmatrix} \hat{e}_y \\ \hat{e}_z \end{pmatrix}$$

$$\left(\frac{d\mathbf{V}}{dt}\right)_{inertial} = \left(\frac{d\mathbf{V}}{dt}\right)_{body} + \sum_{i=1}^{3} V_i \frac{d\hat{e}_i}{dt}$$

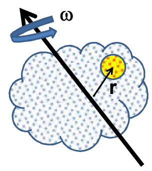
$$\left(\frac{d\mathbf{V}}{dt}\right)_{inertial} = \left(\frac{d\mathbf{V}}{dt}\right)_{body} + \boldsymbol{\omega} \times \mathbf{V}$$

Effects on acceleration:

$$\left(\frac{d}{dt}\frac{d\mathbf{V}}{dt}\right)_{inertial} = \left(\left(\frac{d}{dt}\right)_{body} + \boldsymbol{\omega} \times\right)\left\{\left(\frac{d\mathbf{V}}{dt}\right)_{body} + \boldsymbol{\omega} \times \mathbf{V}\right\}$$

$$\left(\frac{d^2\mathbf{V}}{dt^2}\right)_{inertial} = \left(\frac{d^2\mathbf{V}}{dt^2}\right)_{body} + 2\boldsymbol{\omega} \times \left(\frac{d\mathbf{V}}{dt}\right)_{body} + \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{V} + \boldsymbol{\omega} \times \boldsymbol{\omega} \times \mathbf{V}$$

Kinetic energy of rigid body :



$$\left(\frac{d\mathbf{r}}{dt}\right)_{inertial} = \left(\frac{d\mathbf{r}}{dt}\right)_{body} + \boldsymbol{\omega}\times\mathbf{r}$$

=0 for rigid body

$$\rightarrow \left(\frac{d\mathbf{r}}{dt}\right)_{inertial} = \boldsymbol{\omega}\times\mathbf{r}$$

$$T = \sum_p \frac{1}{2} m_p v_p^2 = \sum_p \frac{1}{2} m_p \left(\left|\boldsymbol{\omega}\times\mathbf{r}_p\right|\right)^2$$

$$= \sum_p \frac{1}{2} m_p \left(\boldsymbol{\omega}\times\mathbf{r}_p\right)\cdot\left(\boldsymbol{\omega}\times\mathbf{r}_p\right)$$

$$= \sum_p \frac{1}{2} m_p \left[(\boldsymbol{\omega}\cdot\boldsymbol{\omega})(\mathbf{r}_p\cdot\mathbf{r}_p) - (\mathbf{r}_p\cdot\boldsymbol{\omega})^2\right]$$

$$T = \sum_p \frac{1}{2} m_p \left[(\boldsymbol{\omega}\cdot\boldsymbol{\omega})(\mathbf{r}_p\cdot\mathbf{r}_p) - (\mathbf{r}_p\cdot\boldsymbol{\omega})^2\right]$$

$$= \frac{1}{2}\boldsymbol{\omega}\cdot\overleftrightarrow{\mathbf{I}}\cdot\boldsymbol{\omega}$$

Moment of inertia tensor :

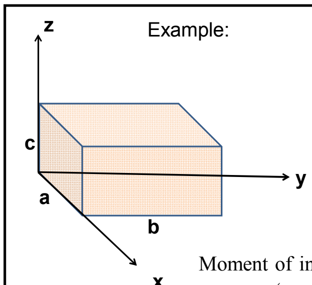
$$\overleftrightarrow{\mathbf{I}} \equiv \sum_p m_p\left(\mathbf{1}r_p^2 - \mathbf{r}_p\mathbf{r}_p\right) \qquad \text{(dyad notation)}$$

Matrix notation :

$$\overleftrightarrow{\mathbf{I}} \equiv \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$$

$$I_{ij} \equiv \sum_p m_p\left(\delta_{ij} r_p^2 - r_{pi} r_{pj}\right)$$

Example:



Moment of inertia tensor :

$$\overleftrightarrow{\mathbf{I}} = M\begin{pmatrix} \frac{1}{3}(b^2+c^2) & -\frac{1}{4}ab & -\frac{1}{4}ac \\ -\frac{1}{4}ab & \frac{1}{3}(a^2+c^2) & -\frac{1}{4}bc \\ -\frac{1}{4}ac & -\frac{1}{4}bc & \frac{1}{3}(a^2+b^2) \end{pmatrix}$$

Properties of moment of inertia tensor:
- Symmetric matrix → real eigenvalues I_1, I_2, I_3
- → orthogonal eigenvectors

$$\ddot{\mathbf{I}} \cdot \hat{\mathbf{e}}_i = I_i \hat{\mathbf{e}}_i \qquad i = 1,2,3$$

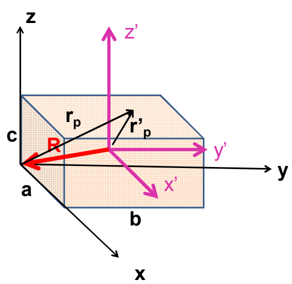
Moment of inertia tensor :

$$\ddot{\mathbf{I}} = M\begin{pmatrix} \frac{1}{3}(b^2+c^2) & -\frac{1}{4}ab & -\frac{1}{4}ac \\ -\frac{1}{4}ab & \frac{1}{3}(a^2+c^2) & -\frac{1}{4}bc \\ -\frac{1}{4}ac & -\frac{1}{4}bc & \frac{1}{3}(a^2+b^2) \end{pmatrix}$$

For $a = b = c$:

$$I_1 = \frac{1}{6}Ma^2 \quad I_2 = \frac{11}{12}Ma^2 \quad I_3 = \frac{11}{12}Ma^2$$

Changing origin of rotation



$$I_{ij} \equiv \sum_p m_p\left(\delta_{ij} r_p^2 - r_{pi} r_{pj}\right)$$

$$I'_{ij} \equiv \sum_p m_p\left(\delta_{ij} r'^2_p - r'_{pi} r'_{pj}\right)$$

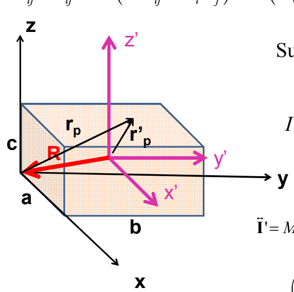
$$\mathbf{r}'_p = \mathbf{r}_p + \mathbf{R}$$

Define the center of mass :

$$\mathbf{r}_{CM} = \frac{\sum_p m_p \mathbf{r}_p}{\sum_p m_p} \equiv \frac{\sum_p m_p \mathbf{r}_p}{M}$$

$$I'_{ij} = I_{ij} + M\left(R^2\delta_{ij} - R_i R_j\right) + M\left(2\mathbf{r}_{CM} \cdot \mathbf{R}\delta_{ij} - r_{CMi} R_j - R_i r_{CMj}\right)$$

$$I'_{ij} = I_{ij} + M\left(R^2\delta_{ij} - R_i R_j\right) + M\left(2\mathbf{r}_{CM} \cdot \mathbf{R}\delta_{ij} - r_{CMi} R_j - R_i r_{CMj}\right)$$

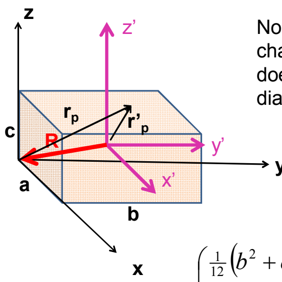


Suppose that $\mathbf{R} = -\frac{a}{2}\hat{\mathbf{x}} - \frac{b}{2}\hat{\mathbf{y}} - \frac{c}{2}\hat{\mathbf{z}}$

and $\mathbf{r}_{CM} = -\mathbf{R}$

$$I'_{ij} = I_{ij} - M\left(R^2\delta_{ij} - R_i R_j\right)$$

$$\ddot{\mathbf{I}}' = M\begin{pmatrix} \frac{1}{3}(b^2+c^2) & -\frac{1}{4}ab & -\frac{1}{4}ac \\ -\frac{1}{4}ab & \frac{1}{3}(a^2+c^2) & -\frac{1}{4}bc \\ -\frac{1}{4}ac & -\frac{1}{4}bc & \frac{1}{3}(a^2+b^2) \end{pmatrix} - M\begin{pmatrix} \frac{1}{4}(b^2+c^2) & -\frac{1}{4}ab & -\frac{1}{4}ac \\ -\frac{1}{4}ab & \frac{1}{4}(a^2+c^2) & -\frac{1}{4}bc \\ -\frac{1}{4}ac & -\frac{1}{4}bc & \frac{1}{4}(a^2+b^2) \end{pmatrix}$$



Note: This is a special case; changing the center of rotation does not necessarily result in a diagonal **I'**

$$\ddot{\mathbf{I}}' = M\begin{pmatrix} \frac{1}{12}\left(b^2+c^2\right) & 0 & 0 \\ 0 & \frac{1}{12}\left(a^2+c^2\right) & 0 \\ 0 & 0 & \frac{1}{12}\left(a^2+b^2\right) \end{pmatrix}$$

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Descriptions of rotation about a given origin

For general coordinate system

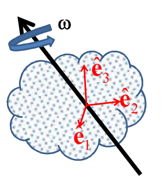
$$T = \frac{1}{2}\sum_{ij} I_{ij}\omega_i\omega_j$$

For (body fixed) coordinate system that diagonalizes moment of inertia tensor :

$$\ddot{\mathbf{I}}\cdot\hat{\mathbf{e}}_i = I_i\hat{\mathbf{e}}_i \qquad i = 1,2,3$$

$$\boldsymbol{\omega} = \tilde{\omega}_1\hat{\mathbf{e}}_1 + \tilde{\omega}_2\hat{\mathbf{e}}_2 + \tilde{\omega}_3\hat{\mathbf{e}}_3$$

$$\Rightarrow T = \frac{1}{2}\sum_i I_i\tilde{\omega}_i^2$$



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Descriptions of rotation about a given origin -- continued

Time rate of change of angular momentum

$$\frac{d\mathbf{L}}{dt} = \left(\frac{d\mathbf{L}}{dt}\right)_{body} + \boldsymbol{\omega}\times\mathbf{L}$$

For (body fixed) coordinate system that diagonalizes moment of inertia tensor :

$$\ddot{\mathbf{I}}\cdot\hat{\mathbf{e}}_i = I_i\hat{\mathbf{e}}_i \qquad \boldsymbol{\omega} = \tilde{\omega}_1\hat{\mathbf{e}}_1 + \tilde{\omega}_2\hat{\mathbf{e}}_2 + \tilde{\omega}_3\hat{\mathbf{e}}_3$$

$$\mathbf{L} = I_1\tilde{\omega}_1\hat{\mathbf{e}}_1 + I_2\tilde{\omega}_2\hat{\mathbf{e}}_2 + I_3\tilde{\omega}_3\hat{\mathbf{e}}_3$$

$$\frac{d\mathbf{L}}{dt} = I_1\dot{\tilde{\omega}}_1\hat{\mathbf{e}}_1 + I_2\dot{\tilde{\omega}}_2\hat{\mathbf{e}}_2 + I_3\dot{\tilde{\omega}}_3\hat{\mathbf{e}}_3 + \tilde{\omega}_2\tilde{\omega}_3\left(I_3 - I_2\right)\hat{\mathbf{e}}_1$$
$$+ \tilde{\omega}_3\tilde{\omega}_1\left(I_1 - I_3\right)\hat{\mathbf{e}}_2 + \tilde{\omega}_1\tilde{\omega}_2\left(I_2 - I_1\right)\hat{\mathbf{e}}_3$$

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Descriptions of rotation about a given origin -- continued

Note that the torque equation

$$\frac{d\mathbf{L}}{dt} = \left(\frac{d\mathbf{L}}{dt}\right)_{body} + \boldsymbol{\omega}\times\mathbf{L} = \boldsymbol{\tau}$$

is very difficult to solve directly in the body fixed frame.

For $\boldsymbol{\tau} = 0$ we can solve the Euler equations :

$$\frac{d\mathbf{L}}{dt} = I_1\dot{\tilde{\omega}}_1\hat{\mathbf{e}}_1 + I_2\dot{\tilde{\omega}}_2\hat{\mathbf{e}}_2 + I_3\dot{\tilde{\omega}}_3\hat{\mathbf{e}}_3 + \tilde{\omega}_2\tilde{\omega}_3(I_3 - I_2)\hat{\mathbf{e}}_1$$
$$+\tilde{\omega}_3\tilde{\omega}_1(I_1 - I_3)\hat{\mathbf{e}}_2 + \tilde{\omega}_1\tilde{\omega}_2(I_2 - I_1)\hat{\mathbf{e}}_3 = 0$$

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Euler equations for rotation in body fixed frame :

$$I_1\dot{\tilde{\omega}}_1 + \tilde{\omega}_2\tilde{\omega}_3(I_3 - I_2) = 0$$
$$I_2\dot{\tilde{\omega}}_2 + \tilde{\omega}_3\tilde{\omega}_1(I_1 - I_3) = 0$$
$$I_3\dot{\tilde{\omega}}_3 + \tilde{\omega}_1\tilde{\omega}_2(I_2 - I_1) = 0$$

→ Solution for symmetric top -- $I_2 = I_1$:

$$I_1\dot{\tilde{\omega}}_1 + \tilde{\omega}_2\tilde{\omega}_3(I_3 - I_1) = 0$$
$$I_1\dot{\tilde{\omega}}_2 + \tilde{\omega}_3\tilde{\omega}_1(I_1 - I_3) = 0$$
$$I_3\dot{\tilde{\omega}}_3 = 0 \qquad \Rightarrow \tilde{\omega}_3 = \text{(constant)}$$

Define : $\Omega \equiv \tilde{\omega}_3\frac{I_3 - I_1}{I_1}$

$$\dot{\tilde{\omega}}_1 = -\tilde{\omega}_2\Omega$$
$$\dot{\tilde{\omega}}_2 = \tilde{\omega}_1\Omega$$

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Solution of Euler equations for a symmetric top -- continued

$$\dot{\tilde{\omega}}_1 = -\tilde{\omega}_2\Omega \qquad \dot{\tilde{\omega}}_2 = \tilde{\omega}_1\Omega$$

where $\Omega \equiv \tilde{\omega}_3\frac{I_3 - I_1}{I_1}$

Solution:

$$\tilde{\omega}_1(t) = A\cos(\Omega t + \phi)$$
$$\tilde{\omega}_2(t) = A\sin(\Omega t + \phi)$$
$$\tilde{\omega}_3(t) = \tilde{\omega}_3 \quad \text{(constant)}$$

$$T = \frac{1}{2}\sum_i I_i\tilde{\omega}_i^2 = \frac{1}{2}I_1A^2 + \frac{1}{2}I_3\tilde{\omega}_3^2$$

$$\mathbf{L} = I_1\tilde{\omega}_1\hat{\mathbf{e}}_1 + I_2\tilde{\omega}_2\hat{\mathbf{e}}_2 + I_3\tilde{\omega}_3\hat{\mathbf{e}}_3$$
$$= I_1A\left(\cos(\Omega t + \varphi)\hat{\mathbf{e}}_1 + \sin(\Omega t + \varphi)\hat{\mathbf{e}}_2\right) + I_3\tilde{\omega}_3\hat{\mathbf{e}}_3$$

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Euler equations for rotation in body fixed frame :

$$I_1\dot{\tilde{\omega}}_1 + \tilde{\omega}_2\tilde{\omega}_3(I_3 - I_2) = 0$$
$$I_2\dot{\tilde{\omega}}_2 + \tilde{\omega}_3\tilde{\omega}_1(I_1 - I_3) = 0$$
$$I_3\dot{\tilde{\omega}}_3 + \tilde{\omega}_1\tilde{\omega}_2(I_2 - I_1) = 0$$

→ Solution for asymmetric top - - $I_3 \neq I_2 \neq I_1$:

$$I_1\dot{\tilde{\omega}}_1 + \tilde{\omega}_2\tilde{\omega}_3(I_3 - I_2) = 0$$
$$I_2\dot{\tilde{\omega}}_2 + \tilde{\omega}_3\tilde{\omega}_1(I_1 - I_3) = 0$$
$$I_3\dot{\tilde{\omega}}_3 + \tilde{\omega}_1\tilde{\omega}_2(I_2 - I_1) = 0$$

Suppose : $\dot{\tilde{\omega}}_3 \approx 0$ Define : $\Omega_1 \equiv \tilde{\omega}_3 \frac{I_3 - I_2}{I_1}$

Define : $\Omega_2 \equiv \tilde{\omega}_3 \frac{I_3 - I_1}{I_2}$

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Euler equations for rotation in body fixed frame :

$$I_1\dot{\tilde{\omega}}_1 + \tilde{\omega}_2\tilde{\omega}_3(I_3 - I_2) = 0$$
$$I_2\dot{\tilde{\omega}}_2 + \tilde{\omega}_3\tilde{\omega}_1(I_1 - I_3) = 0$$
$$I_3\dot{\tilde{\omega}}_3 + \tilde{\omega}_1\tilde{\omega}_2(I_2 - I_1) = 0$$

Solution for asymmetric top -- $I_3 \neq I_2 \neq I_1$:

Approximate solution --

Suppose: $\dot{\tilde{\omega}}_3 \approx 0$ Define: $\Omega_1 \equiv \tilde{\omega}_3 \frac{I_3 - I_2}{I_1}$

Define: $\Omega_2 \equiv \tilde{\omega}_3 \frac{I_3 - I_1}{I_2}$

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Euler equations for asymmetric top -- continued

$$I_1\dot{\tilde{\omega}}_1 + \tilde{\omega}_2\tilde{\omega}_3(I_3 - I_2) = 0$$
$$I_2\dot{\tilde{\omega}}_2 + \tilde{\omega}_3\tilde{\omega}_1(I_1 - I_3) = 0$$
$$I_3\dot{\tilde{\omega}}_3 + \tilde{\omega}_1\tilde{\omega}_2(I_2 - I_1) = 0$$

If $\dot{\tilde{\omega}}_3 \approx 0$, Define: $\Omega_1 \equiv \tilde{\omega}_3 \frac{I_3 - I_2}{I_1}$ $\Omega_2 \equiv \tilde{\omega}_3 \frac{I_3 - I_1}{I_2}$

$$\dot{\tilde{\omega}}_1 = -\Omega_1\tilde{\omega}_2 \qquad \dot{\tilde{\omega}}_2 = \Omega_2\tilde{\omega}_1$$

If Ω_1 and Ω_2 are both positive or both negative :

$$\tilde{\omega}_1(t) \approx A\cos\left(\sqrt{\Omega_1\Omega_2}\,t + \varphi\right)$$
$$\tilde{\omega}_2(t) \approx A\sqrt{\frac{\Omega_2}{\Omega_1}}\sin\left(\sqrt{\Omega_1\Omega_2}\,t + \varphi\right)$$

$\Rightarrow$ If Ω_1 and Ω_2 have opposite signs, solution is unstable.

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