PHY 711 Classical Mechanics and Mathematical Methods
9-9:50 AM MWF Olin 107

Plan for Lecture 18:

Rotational motion (Chapter 5)

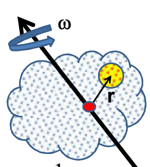
1. **Rigid body motion in body fixed frame**
2. **Conversion between body and inertial reference frames**
3. **Symmetric top motion**

Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W Reading	Topic	Assignment	Due
1	Mon, 8/28/2017	Chap. 1	Introduction	#1	9/6/2017
2	Wed, 8/30/2017	Chap. 1	Scattering theory	#2	9/6/2017
3	Fri, 9/01/2017	Chap. 1	Scattering theory		
7	Mon, 9/11/2017	Chap.3	Calculus of variations	#6	9/13/2017
8	Wed, 9/13/2017	Chap. 3	Lagrangian Mechanics	#7	9/15/2017
9	Fri, 9/15/2017	Chap. 3 and 6	Lagrangian mechanics and constraints	#8	9/20/2017
10	Mon, 9/18/2017	Chap. 3 and 6	Constants of the motion		
11	Wed, 9/20/2017	Chap. 3 and 6	Hamiltonian formalism	#9	9/27/2017
12	Fri, 9/22/2017	Chap. 3 and 6	Liouville equation		
13	Mon, 9/25/2017	Chap. 6	Canonical transformations		
14	Wed, 9/27/2017	Chap. 4	Motion from Small oscillations about equilibrium		
15	Fri, 9/29/2017	Chap. 1-4,6	Review		
	Mon, 10/02/2017		*Take-home exam -- No class*		
	Wed, 10/04/2017		*Take-home exam -- No class*		
16	Fri, 10/06/2017	Chap. 4	Vibrational modes		
17	Mon, 10/09/2017	Chap. 5	Rigid body motion	#10	10/16/2017
18	Wed, 10/11/2017	Chap. 5	Rigid body motion		
	Fri, 10/13/2017		*Fall break -- No class*		
19	Mon, 10/16/2017				
20	Wed, 10/18/2017				
21	Fri, 10/20/2017				
22	Mon, 10/23/2017				

Summary of previous results describing rigid bodies rotating about a fixed origin



$$\left(\frac{d\mathbf{r}}{dt}\right)_{inertial} = \boldsymbol{\omega}\times\mathbf{r}$$

Kinetic energy: $$T = \sum_p \frac{1}{2} m_p v_p^2 = \sum_p \frac{1}{2} m_p \left(\left|\boldsymbol{\omega}\times\mathbf{r}_p\right|\right)^2$$

$$= \sum_p \frac{1}{2} m_p \left(\boldsymbol{\omega}\times\mathbf{r}_p\right)\cdot\left(\boldsymbol{\omega}\times\mathbf{r}_p\right)$$

$$= \sum_p \frac{1}{2} m_p \left[\left(\boldsymbol{\omega}\cdot\boldsymbol{\omega}\right)\left(\mathbf{r}_p\cdot\mathbf{r}_p\right) - \left(\mathbf{r}_p\cdot\boldsymbol{\omega}\right)^2\right]$$

$$= \boldsymbol{\omega}\cdot\overleftrightarrow{\mathbf{I}}\cdot\boldsymbol{\omega} \qquad \overleftrightarrow{\mathbf{I}} \equiv \sum_p m_p\left(\mathbf{1}r_p^2 - \mathbf{r}_p\mathbf{r}_p\right)$$

Moment of inertia tensor

Matrix notation:

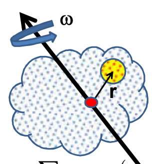
$$\overleftrightarrow{\mathbf{I}} \equiv \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \qquad I_{ij} \equiv \sum_p m_p \left(\delta_{ij} r_p^2 - r_{pi} r_{pj} \right)$$

For general coordinate system: $T = \frac{1}{2} \sum_{ij} I_{ij} \omega_i \omega_j$

For (body fixed) coordinate system that diagonalizes moment of inertia tensor: $\overleftrightarrow{\mathbf{I}} \cdot \hat{\mathbf{e}}_i = I_i \hat{\mathbf{e}}_i \qquad i = 1,2,3$

$$\boldsymbol{\omega} = \tilde{\omega}_1 \hat{\mathbf{e}}_1 + \tilde{\omega}_2 \hat{\mathbf{e}}_2 + \tilde{\omega}_3 \hat{\mathbf{e}}_3 \qquad \Rightarrow T = \frac{1}{2} \sum_i I_i \tilde{\omega}_i^2$$

Continued -- summary of previous results describing rigid bodies rotating about a fixed origin



$$\left(\frac{d\mathbf{r}}{dt} \right)_{inertial} = \boldsymbol{\omega} \times \mathbf{r}$$

Angular momentum: $\mathbf{L} = \sum_p m_p \mathbf{r}_p \times \mathbf{v}_p = \sum_p m_p \mathbf{r}_p \times \left(\boldsymbol{\omega} \times \mathbf{r}_p \right)$

$$= \sum_p m_p \left[\boldsymbol{\omega} \left(\mathbf{r}_p \cdot \mathbf{r}_p \right) - \mathbf{r}_p \left(\mathbf{r}_p \cdot \boldsymbol{\omega} \right) \right]$$

$$= \overleftrightarrow{\mathbf{I}} \cdot \boldsymbol{\omega} \qquad \overleftrightarrow{\mathbf{I}} \equiv \sum_p m_p \left(\mathbf{1} r_p^2 - \mathbf{r}_p \mathbf{r}_p \right)$$

Descriptions of rotation about a given origin -- continued

For (body fixed) coordinate system that diagonalizes moment of inertia tensor:

$$\overleftrightarrow{\mathbf{I}} \cdot \hat{\mathbf{e}}_i = I_i \hat{\mathbf{e}}_i \qquad \boldsymbol{\omega} = \tilde{\omega}_1 \hat{\mathbf{e}}_1 + \tilde{\omega}_2 \hat{\mathbf{e}}_2 + \tilde{\omega}_3 \hat{\mathbf{e}}_3$$

$$\mathbf{L} = I_1 \tilde{\omega}_1 \hat{\mathbf{e}}_1 + I_2 \tilde{\omega}_2 \hat{\mathbf{e}}_2 + I_3 \tilde{\omega}_3 \hat{\mathbf{e}}_3$$

Time derivative: $\frac{d\mathbf{L}}{dt} = \left(\frac{d\mathbf{L}}{dt} \right)_{body} + \boldsymbol{\omega} \times \mathbf{L}$

$$\frac{d\mathbf{L}}{dt} = I_1 \dot{\tilde{\omega}}_1 \hat{\mathbf{e}}_1 + I_2 \dot{\tilde{\omega}}_2 \hat{\mathbf{e}}_2 + I_3 \dot{\tilde{\omega}}_3 \hat{\mathbf{e}}_3 + \tilde{\omega}_2 \tilde{\omega}_3 \left(I_3 - I_2 \right) \hat{\mathbf{e}}_1 + \tilde{\omega}_3 \tilde{\omega}_1 \left(I_1 - I_3 \right) \hat{\mathbf{e}}_2 + \tilde{\omega}_1 \tilde{\omega}_2 \left(I_2 - I_1 \right) \hat{\mathbf{e}}_3$$

Descriptions of rotation about a given origin -- continued

Note that the torque equation

$$\frac{d\mathbf{L}}{dt} = \left(\frac{d\mathbf{L}}{dt}\right)_{body} + \boldsymbol{\omega}\times\mathbf{L} = \boldsymbol{\tau}$$

is very difficult to solve directly in the body fixed frame.

For $\boldsymbol{\tau} = 0$ we can solve the Euler equations:

$$\frac{d\mathbf{L}}{dt} = 0 = I_1\dot{\tilde{\omega}}_1\hat{\mathbf{e}}_1 + I_2\dot{\tilde{\omega}}_2\hat{\mathbf{e}}_2 + I_3\dot{\tilde{\omega}}_3\hat{\mathbf{e}}_3 +$$
$$\tilde{\omega}_2\tilde{\omega}_3\left(I_3 - I_2\right)\hat{\mathbf{e}}_1 + \tilde{\omega}_3\tilde{\omega}_1\left(I_1 - I_3\right)\hat{\mathbf{e}}_2 + \tilde{\omega}_1\tilde{\omega}_2\left(I_2 - I_1\right)\hat{\mathbf{e}}_3$$

$$I_1\dot{\tilde{\omega}}_1 + \tilde{\omega}_2\tilde{\omega}_3\left(I_3 - I_2\right) = 0$$
$$I_2\dot{\tilde{\omega}}_2 + \tilde{\omega}_3\tilde{\omega}_1\left(I_1 - I_3\right) = 0$$
$$I_3\dot{\tilde{\omega}}_3 + \tilde{\omega}_1\tilde{\omega}_2\left(I_2 - I_1\right) = 0$$

Euler equations for rotation in body fixed frame :

$$I_1\dot{\tilde{\omega}}_1 + \tilde{\omega}_2\tilde{\omega}_3\left(I_3 - I_2\right) = 0$$
$$I_2\dot{\tilde{\omega}}_2 + \tilde{\omega}_3\tilde{\omega}_1\left(I_1 - I_3\right) = 0$$
$$I_3\dot{\tilde{\omega}}_3 + \tilde{\omega}_1\tilde{\omega}_2\left(I_2 - I_1\right) = 0$$

Solution for symmetric top -- $I_2 = I_1$:

$$I_1\dot{\tilde{\omega}}_1 + \tilde{\omega}_2\tilde{\omega}_3\left(I_3 - I_1\right) = 0$$
$$I_1\dot{\tilde{\omega}}_2 + \tilde{\omega}_3\tilde{\omega}_1\left(I_1 - I_3\right) = 0$$
$$I_3\dot{\tilde{\omega}}_3 = 0 \qquad \Rightarrow \tilde{\omega}_3 = \text{(constant)}$$

Define : $\Omega \equiv \tilde{\omega}_3 \dfrac{I_3 - I_1}{I_1}$

$$\dot{\tilde{\omega}}_1 = -\tilde{\omega}_2\Omega$$
$$\dot{\tilde{\omega}}_2 = \tilde{\omega}_1\Omega$$

Solution of Euler equations for a symmetric top -- continued

$$\dot{\tilde{\omega}}_1 = -\tilde{\omega}_2\Omega \qquad \dot{\tilde{\omega}}_2 = \tilde{\omega}_1\Omega$$

where $\Omega \equiv \tilde{\omega}_3 \dfrac{I_3 - I_1}{I_1}$

Solution : $\tilde{\omega}_1(t) = A\cos(\Omega t + \varphi)$

$\tilde{\omega}_2(t) = A\sin(\Omega t + \varphi)$

$$T = \frac{1}{2}\sum_i I_i\tilde{\omega}_i^2 = \frac{1}{2}I_1A^2 + \frac{1}{2}I_3\tilde{\omega}_3^2$$

$$\mathbf{L} = I_1\tilde{\omega}_1\hat{\mathbf{e}}_1 + I_2\tilde{\omega}_2\hat{\mathbf{e}}_2 + I_3\tilde{\omega}_3\hat{\mathbf{e}}_3$$
$$= I_1A\left(\cos(\Omega t + \varphi)\hat{\mathbf{e}}_1 + \sin(\Omega t + \varphi)\hat{\mathbf{e}}_2\right) + I_3\tilde{\omega}_3\hat{\mathbf{e}}_3$$

Euler equations for rotation in body fixed frame :

$$I_1\dot{\tilde{\omega}}_1 + \tilde{\omega}_2\tilde{\omega}_3(I_3 - I_2) = 0$$
$$I_2\dot{\tilde{\omega}}_2 + \tilde{\omega}_3\tilde{\omega}_1(I_1 - I_3) = 0$$
$$I_3\dot{\tilde{\omega}}_3 + \tilde{\omega}_1\tilde{\omega}_2(I_2 - I_1) = 0$$

Solution for asymmetric top -- $I_3 \neq I_2 \neq I_1$:

Suppose : $\dot{\tilde{\omega}}_3 \approx 0$ Define : $\Omega_1 \equiv \tilde{\omega}_3 \dfrac{I_3 - I_2}{I_1}$

Define : $\Omega_2 \equiv \tilde{\omega}_3 \dfrac{I_3 - I_1}{I_2}$

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Euler equations for asymmetric top -- continued

$$I_1\dot{\tilde{\omega}}_1 + \tilde{\omega}_2\tilde{\omega}_3(I_3 - I_2) = 0$$
$$I_2\dot{\tilde{\omega}}_2 + \tilde{\omega}_3\tilde{\omega}_1(I_1 - I_3) = 0$$
$$I_3\dot{\tilde{\omega}}_3 + \tilde{\omega}_1\tilde{\omega}_2(I_2 - I_1) = 0$$

If $\dot{\tilde{\omega}}_3 \approx 0$, Define: $\Omega_1 \equiv \tilde{\omega}_3 \dfrac{I_3 - I_2}{I_1}$ $\Omega_2 \equiv \tilde{\omega}_3 \dfrac{I_3 - I_1}{I_2}$

$$\dot{\tilde{\omega}}_1 = -\Omega_1\tilde{\omega}_2 \qquad \dot{\tilde{\omega}}_2 = \Omega_2\tilde{\omega}_1$$

If Ω_1 and Ω_2 are both positive or both negative :

$$\tilde{\omega}_1(t) \approx A\cos\left(\sqrt{\Omega_1\Omega_2}\,t + \varphi\right)$$
$$\tilde{\omega}_2(t) \approx A\sqrt{\frac{\Omega_2}{\Omega_1}}\sin\left(\sqrt{\Omega_1\Omega_2}\,t + \varphi\right)$$

$\Rightarrow$ If Ω_1 and Ω_2 have opposite signs, solution is unstable.

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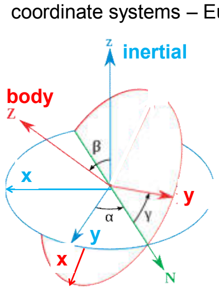
Transformation between body-fixed and inertial coordinate systems – Euler angles



http://en.wikipedia.org/wiki/Euler_angles

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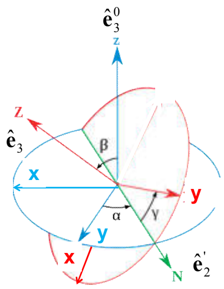
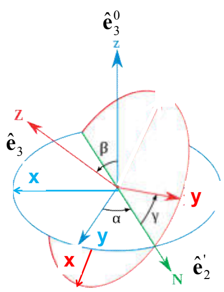


$$\tilde{\boldsymbol{\omega}} = \dot{\alpha}\,\hat{\mathbf{e}}_3^0 + \dot{\beta}\,\hat{\mathbf{e}}_2^{'} + \dot{\gamma}\,\hat{\mathbf{e}}_3$$

Need to express all components in body-fixed frame:

$$\tilde{\boldsymbol{\omega}} = \tilde{\omega}_1\hat{\mathbf{e}}_1 + \tilde{\omega}_2\hat{\mathbf{e}}_2 + \tilde{\omega}_3\hat{\mathbf{e}}_3$$

$$\tilde{\boldsymbol{\omega}} = \dot{\alpha}\,\hat{\mathbf{e}}_3^0 + \dot{\beta}\,\hat{\mathbf{e}}_2^{'} + \dot{\gamma}\,\hat{\mathbf{e}}_3$$

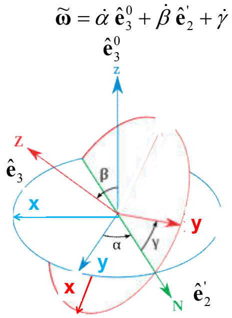


$$\hat{\mathbf{e}}_2^{'} = \sin\gamma\,\hat{\mathbf{e}}_1 + \cos\gamma\,\hat{\mathbf{e}}_2$$

Matrix representation :

$$\hat{\mathbf{e}}_2^{'} = \begin{pmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \sin\gamma \\ \cos\gamma \\ 0 \end{pmatrix}$$

$$\tilde{\boldsymbol{\omega}} = \dot{\alpha}\,\hat{\mathbf{e}}_3^0 + \dot{\beta}\,\hat{\mathbf{e}}_2^{'} + \dot{\gamma}\,\hat{\mathbf{e}}_3$$



$$\hat{\mathbf{e}}_3^0 = -\sin\beta\,\hat{\mathbf{e}}'_1 + \cos\beta\,\hat{\mathbf{e}}'_3$$
$$= -\cos\gamma\sin\beta\,\hat{\mathbf{e}}_1 + \sin\gamma\sin\beta\,\hat{\mathbf{e}}_2 + \cos\beta\,\hat{\mathbf{e}}_3$$

Matrix representation:

$$\hat{\mathbf{e}}_3^0 = \begin{pmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$
$$= \begin{pmatrix} -\sin\beta\cos\gamma \\ \sin\beta\sin\gamma \\ \cos\beta \end{pmatrix}$$

$$\tilde{\boldsymbol{\omega}} = \dot{\alpha}\,\hat{\mathbf{e}}_3^0 + \dot{\beta}\,\hat{\mathbf{e}}_2^{'} + \dot{\gamma}\,\hat{\mathbf{e}}_3$$

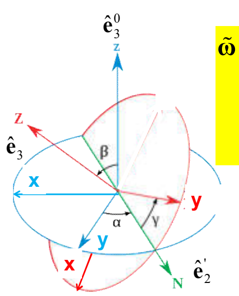
$$\tilde{\boldsymbol{\omega}} = \dot{\alpha}\begin{pmatrix} -\sin\beta\cos\gamma \\ \sin\beta\sin\gamma \\ \cos\beta \end{pmatrix} + \dot{\beta}\begin{pmatrix} \sin\gamma \\ \cos\gamma \\ 0 \end{pmatrix} + \dot{\gamma}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\tilde{\boldsymbol{\omega}} = \tilde{\omega}_1\hat{\mathbf{e}}_1 + \tilde{\omega}_2\hat{\mathbf{e}}_2 + \tilde{\omega}_3\hat{\mathbf{e}}_3$$

$$\tilde{\boldsymbol{\omega}} = \dot{\alpha}\begin{pmatrix} -\sin\beta\cos\gamma \\ \sin\beta\sin\gamma \\ \cos\beta \end{pmatrix} + \dot{\beta}\begin{pmatrix} \sin\gamma \\ \cos\gamma \\ 0 \end{pmatrix} + \dot{\gamma}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\tilde{\omega}_1 = \dot{\alpha}(-\sin\beta\cos\gamma) + \dot{\beta}\sin\gamma$$
$$\tilde{\omega}_2 = \dot{\alpha}(\sin\beta\sin\gamma) + \dot{\beta}\cos\gamma$$
$$\tilde{\omega}_3 = \dot{\alpha}\cos\beta + \dot{\gamma}$$

$$\tilde{\boldsymbol{\omega}} = \dot{\alpha}\,\hat{\mathbf{e}}_3^0 + \dot{\beta}\,\hat{\mathbf{e}}_2^{'} + \dot{\gamma}\,\hat{\mathbf{e}}_3$$



$$\begin{aligned}\tilde{\boldsymbol{\omega}} &= \left[\dot{\alpha}\left(-\sin\beta\cos\gamma\right) + \dot{\beta}\sin\gamma\right]\hat{\mathbf{e}}_1 \\ &+ \left[\dot{\alpha}\left(\sin\beta\sin\gamma\right) + \dot{\beta}\cos\gamma\right]\hat{\mathbf{e}}_2 \\ &+ \left[\dot{\alpha}\cos\beta + \dot{\gamma}\right]\hat{\mathbf{e}}_3\end{aligned}$$

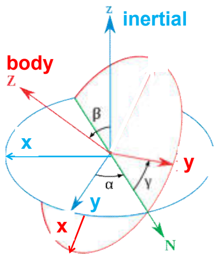
Rotational kinetic energy

$$\begin{aligned}T(\alpha,\beta,\gamma,\dot{\alpha},\dot{\beta},\dot{\gamma}) &= \frac{1}{2}I_1\tilde{\omega}_1^2 + \frac{1}{2}I_2\tilde{\omega}_2^2 + \frac{1}{2}I_3\tilde{\omega}_3^2 \\ &= \frac{1}{2}I_1\left[\dot{\alpha}(-\sin\beta\cos\gamma) + \dot{\beta}\sin\gamma\right]^2 \\ &+ \frac{1}{2}I_2\left[\dot{\alpha}(\sin\beta\sin\gamma) + \dot{\beta}\cos\gamma\right]^2 \\ &+ \frac{1}{2}I_3\left[\dot{\alpha}\cos\beta + \dot{\gamma}\right]^2\end{aligned}$$

If $I_1 = I_2$:

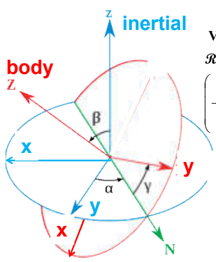
$$T(\alpha,\beta,\gamma,\dot{\alpha},\dot{\beta},\dot{\gamma}) = \frac{1}{2}I_1\left(\dot{\alpha}^2\sin^2\beta + \dot{\beta}^2\right) + \frac{1}{2}I_3\left[\dot{\alpha}\cos\beta + \dot{\gamma}\right]^2$$

Transformation between body-fixed and inertial coordinate systems – Euler angles



http://en.wikipedia.org/wiki/Euler_angles

General transformation between rotated coordinates – Euler angles

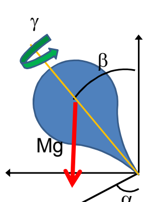


$$\mathbf{V}' = \mathcal{R}\mathbf{V} = \mathcal{R}_\alpha \mathcal{R}_\beta \mathcal{R}_\gamma \mathbf{V}$$

$$\mathcal{R} = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

http://en.wikipedia.org/wiki/Euler_angles

Motion of a symmetric top under the influence of the torque of gravity:



$$L(\alpha,\beta,\gamma,\dot\alpha,\dot\beta,\dot\gamma) = \frac{1}{2}I_1\left(\dot\alpha^2\sin^2\beta + \dot\beta^2\right) + \frac{1}{2}I_3\left[\dot\alpha\cos\beta + \dot\gamma\right]^2 - Mgl\cos\beta$$

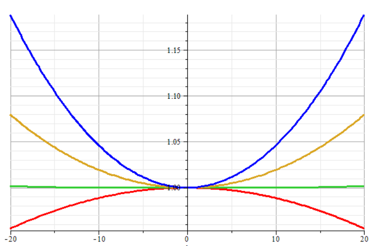
$$L(\alpha,\beta,\gamma,\dot{\alpha},\dot{\beta},\dot{\gamma})=\frac{1}{2}I_1\left(\dot{\alpha}^2\sin^2\beta+\dot{\beta}^2\right)+$$
$$\frac{1}{2}I_3[\dot{\alpha}\cos\beta+\dot{\gamma}]^2-Mgl\cos\beta$$

Constants of the motion :

$$p_\alpha=\frac{\partial L}{\partial\dot{\alpha}}=I_1\dot{\alpha}\sin^2\beta+I_3[\dot{\alpha}\cos\beta+\dot{\gamma}]\cos\beta$$
$$p_\gamma=\frac{\partial L}{\partial\dot{\gamma}}=I_3[\dot{\alpha}\cos\beta+\dot{\gamma}]$$
$$E=\frac{1}{2}I_1\dot{\beta}^2+\frac{p_\gamma^2}{2I_3}+V_{eff}(\beta)$$
$$L(\beta,\dot{\beta})=\frac{1}{2}I_1\dot{\beta}^2+\frac{(p_\alpha-p_\gamma\cos\beta)^2}{2I_1\sin^2\beta}+\frac{p_\gamma^2}{2I_3}-Mgl\cos\beta$$
$$V_{eff}(\beta)=\frac{(p_\alpha-p_\gamma\cos\beta)^2}{2I_1\sin^2\beta}+Mgl\cos\beta$$

$$E=\frac{1}{2}I_1\dot{\beta}^2+\frac{p_\gamma^2}{2I_3}+\frac{(p_\alpha-p_\gamma\cos\beta)^2}{2I_1\sin^2\beta}+Mgl\cos\beta$$
$$E'=E-\frac{p_\gamma^2}{2I_3}=\frac{1}{2}I_1\dot{\beta}^2+\frac{(p_\alpha-p_\gamma\cos\beta)^2}{2I_1\sin^2\beta}+Mgl\cos\beta$$

Stable/unstable solutions near β=0

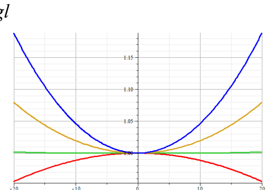


Suppose $p_\alpha=p_\gamma$ and $\beta\approx 0$

$$E'=E-\frac{p_\gamma^2}{2I_3}=\frac{1}{2}I_1\dot{\beta}^2+\frac{(p_\alpha-p_\gamma\cos\beta)^2}{2I_1\sin^2\beta}+Mgl\cos\beta$$
$$E'\approx\frac{1}{2}I_1\dot{\beta}^2+\frac{p_\gamma{}^2}{2I_1}\frac{\left(1-1+\frac{1}{2}\beta^2\right)^2}{\beta^2}+Mgl\left(1-\frac{1}{2}\beta^2\right)$$
$$\approx\frac{1}{2}I_1\dot{\beta}^2+\left(\frac{p_\gamma{}^2}{8I_1}-\frac{Mgl}{2}\right)\beta^2+Mgl$$

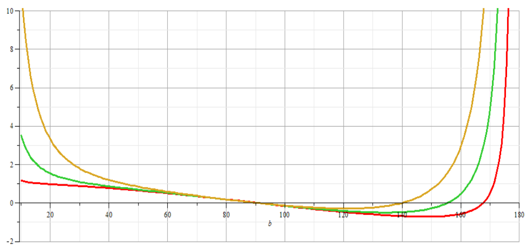
$\Rightarrow$ Stable solution if

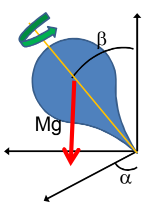
$$p_\gamma\geq\sqrt{4MglI_1}$$



More general case:

$$E' = E - \frac{p_\gamma^2}{2I_3} = \frac{1}{2} I_1 \dot{\beta}^2 + \frac{(p_\alpha - p_\gamma \cos\beta)^2}{2I_1 \sin^2\beta} + Mgl\cos\beta$$





Constants of the motion :

$$p_\gamma = \frac{\partial L}{\partial \dot{\gamma}} = I_3[\dot{\alpha}\cos\beta + \dot{\gamma}]$$

$$p_\alpha = \frac{\partial L}{\partial \dot{\alpha}} = I_1\dot{\alpha}\sin^2\beta + I_3[\dot{\alpha}\cos\beta + \dot{\gamma}]\cos\beta$$

$$= I_1\dot{\alpha}\sin^2\beta + p_\gamma\cos\beta$$

$$E' = E - \frac{p_\gamma^2}{2I_3} = \frac{1}{2} I_1 \dot{\beta}^2 + \frac{(p_\alpha - p_\gamma \cos\beta)^2}{2I_1 \sin^2\beta} + Mgl\cos\beta$$