PHY 711 Classical Mechanics and Mathematical Methods
9-9:50 AM MWF Olin 107

Plan for Lecture 21:

Read Chapter 7 & Appendices A-D

Generalization of the one dimensional wave equation ➔ various mathematical problems and techniques including:

1. **Sturm-Liouville equations**
2. **Orthogonal function expansions; Fourier analysis**
3. **Green's functions methods**
4. **Laplace transformation**
5. **Contour integration methods**

	Mon, 10/02/2017		*Take-home exam -- No class*		
	Wed, 10/04/2017		*Take-home exam -- No class*		
16	Fri, 10/06/2017	Chap. 4	Vibrational modes		
17	Mon, 10/09/2017	Chap. 5	Rigid body motion	#10	10/16/2017
18	Wed, 10/11/2017	Chap. 5	Rigid body motion		
	Fri, 10/13/2017		*Fall break -- No class*		
19	Mon, 10/16/2017		Discuss exam questions and topics for presentations	Topic	10/18/2017
20	Wed, 10/18/2017	Chap. 7	Wave equation in one dimension	#11	10/20/2017
21	Fri, 10/20/2017	Chap. 7	Solutions of Sturm-Liouville equations	#12	10/25/2017
22	Mon, 10/23/2017				
23	Wed, 10/25/2017				
24	Fri, 10/27/2017				
25	Mon, 10/30/2017				
26	Wed, 11/01/2017				
27	Fri, 11/03/2017				
28	Mon, 11/06/2017				
29	Wed, 11/08/2017				
30	Fri, 11/10/2017				

One dimensional wave equation for $\mu(x,t)$:

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0 \quad \text{where } c^2 = \frac{\tau}{\sigma}$$

Generalization for spacially dependent tension and mass density plus an extra potential energy density:

$$\sigma(x)\frac{\partial^2 \mu}{\partial t^2} - \frac{\partial}{\partial x}\left(\tau(x)\frac{\partial \mu}{\partial x}\right) + v(x)\mu = 0$$

Separating time and spatial variables:

$$\mu(x,t) = \rho(x)\cos(\omega t + \phi)$$

Sturm-Liouville equation for spatial function:

$$-\frac{d}{dx}\left(\tau(x)\frac{d\rho}{dx}\right) + v(x)\rho = \omega^2 \sigma(x)\rho$$

Linear second-order ordinary differential equations
Sturm-Liouville equations

Inhomogenous problem: $\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)\varphi(x)=F(x)$

given functions

applied force

solution to be determined

Homogenous problem: *F(x)=0*

Examples of Sturm-Liouville eigenvalue equations --

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)\varphi(x)=0$$

Bessel functions:

$$\tau(x)=-x \quad v(x)=x \quad \sigma(x)=\frac{1}{x} \quad \lambda=\nu^2 \quad \varphi(x)=J_\nu(x)$$

Legendre functions:

$$\tau(x)=-\left(1-x^2\right) \quad v(x)=0 \quad \sigma(x)=1 \quad \lambda=l(l+1) \quad \varphi(x)=P_l(x)$$

Fourier functions:

$$\tau(x)=1 \quad v(x)=0 \quad \sigma(x)=1 \quad \lambda=n^2\pi^2 \quad \varphi(x)=\sin(n\pi x)$$

Solution methods of Sturm-Liouville equations
(assume all functions and constants are real):

Homogenous problem: $\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)\phi_0(x)=0$

Inhomogenous problem: $\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)\phi(x)=F(x)$

Eigenfunctions:

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)\right)f_n(x)=\lambda_n\sigma(x)f_n(x)$$

Orthogonality of eigenfunctions: $\int_a^b \sigma(x)f_n(x)f_m(x)dx=\delta_{nm}N_n$,

where $N_n \equiv \int_a^b \sigma(x)(f_n(x))^2dx$.

Completeness of eigenfunctions:

$$\sigma(x)\sum_n \frac{f_n(x)f_n(x')}{N_n}=\delta(x-x')$$

Comment on "completeness"

It can be shown that for any reasonable function *h(x)*, defined within the interval $a < x < b$, we can expand that function as a linear combination of the eigenfunctions $f_n(x)$

$$h(x) \approx \sum_n C_n f_n(x),$$

$$\text{where} \quad C_n = \frac{1}{N_n}\int_a^b \sigma(x')h(x')f_n(x')dx'.$$

These ideas lead to the notion that the set of eigenfunctions $f_n(x)$ form a ``complete'' set in the sense of ``spanning'' the space of all functions in the interval $a < x < b$, as summarized by the statement:

$$\sigma(x)\sum_n \frac{f_n(x)f_n(x')}{N_n} = \delta(x-x').$$

Variation approximation to lowest eigenvalue

In general, there are several techniques to determine the eigenvalues λ_n and eigenfunctions $f_n(x)$. When it is not possible to find the ``exact'' functions, there are several powerful approximation techniques. For example, the lowest eigenvalue can be approximated by minimizing the function

$$\lambda_0 \le \frac{\langle \tilde{h}|S|\tilde{h}\rangle}{\langle \tilde{h}|\sigma|\tilde{h}\rangle}, \qquad S(x) \equiv -\frac{d}{dx}\tau(x)\frac{d}{dx} + v(x)$$

where $\tilde{h}(x)$ is a variable function which satisfies the correct boundary values. The ``proof'' of this inequality is based on the notion that $\tilde{h}(x)$ can in principle be expanded in terms of the (unknown) exact eigenfunctions $f_n(x)$: $\tilde{h}(x) = \sum_n C_n f_n(x)$, where the coefficients C_n can be assumed to be real.

Estimation of the lowest eigenvalue – continued:

From the eigenfunction equation, we know that

$$S(x)\tilde{h}(x) = S(x)\sum_n C_n f_n(x) = \sum_n C_n \lambda_n \sigma(x) f_n(x).$$

It follows that:

$$\langle \tilde{h}|S|\tilde{h}\rangle = \int_a^b \tilde{h}(x)S(x)\tilde{h}(x)dx = \sum_n |C_n|^2 N_n \lambda_n.$$

It also follows that:

$$\langle \tilde{h}|\sigma|\tilde{h}\rangle = \int_a^b \tilde{h}(x)\sigma(x)\tilde{h}(x)dx = \sum_n |C_n|^2 N_n,$$

$$\text{Therefore} \quad \frac{\langle \tilde{h}|S|\tilde{h}\rangle}{\langle \tilde{h}|\sigma|\tilde{h}\rangle} = \frac{\sum_n |C_n|^2 N_n \lambda_n}{\sum_n |C_n|^2 N_n} \ge \lambda_0.$$

Rayleigh-Ritz method of estimating the lowest eigenvalue

$$\lambda_0 \leq \frac{\langle \tilde{h} | S | \tilde{h} \rangle}{\langle \tilde{h} | \sigma | \tilde{h} \rangle},$$

Example: $-\frac{d^2}{dx^2} f_n(x) = \lambda_n f_n(x)$ with $f_n(0) = f_n(a) = 0$

trial function $f_{\text{trial}}(x) = x(x-a)$

Exact value of $\lambda_0 = \frac{\pi^2}{a^2} = \frac{9.869604404}{a^2}$

Raleigh-Ritz estimate: $\frac{\langle x(a-x) | -\frac{d^2}{dx^2} | x(a-x) \rangle}{\langle x(a-x) | x(a-x) \rangle} = \frac{10}{a^2}$

Green's function solution methods

Suppose that we can find a Green's function defined as follows:

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx} + v(x) - \lambda\sigma(x)\right) G_\lambda(x,x') = \delta(x-x')$$

Recall: Completeness of eigenfunctions:

$$\sigma(x)\sum_n \frac{f_n(x) f_n(x')}{N_n} = \delta(x-x')$$

In terms of eigenfunctions:

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx} + v(x) - \lambda\sigma(x)\right) G_\lambda(x,x') = \sigma(x)\sum_n \frac{f_n(x) f_n(x')}{N_n}$$

$$\Rightarrow G_\lambda(x,x') = \sum_n \frac{f_n(x) f_n(x')/N_n}{\lambda_n - \lambda}$$

Solution to inhomogeneous problem by using Green's functions

Inhomogenous problem:

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx} + v(x) - \lambda\sigma(x)\right)\varphi(x) = F(x)$$

Green's function :

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx} + v(x) - \lambda\sigma(x)\right) G_\lambda(x,x') = \delta(x-x')$$

Formal solution:

$$\varphi_\lambda(x) = \varphi_{\lambda 0}(x) + \int_0^L G_\lambda(x,x') F(x') dx'$$

Solution to homogeneous problem

Example Sturm-Liouville problem:

Example: $\tau(x)=1;\quad \sigma(x)=1;\quad v(x)=0;\quad a=0$ and $b=L$

$\lambda=1;\quad F(x)=F_0\sin\left(\frac{\pi x}{L}\right)$

Inhomogenous equation:

$$\left(-\frac{d^2}{dx^2}-1\right)\phi(x)=F_0\sin\left(\frac{\pi x}{L}\right)$$

Eigenvalue equation:

$$\left(-\frac{d^2}{dx^2}\right)f_n(x)=\lambda_n f_n(x)$$

Eigenfunctions

$$f_n(x)=\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)$$

Eigenvalues:

$$\lambda_n=\left(\frac{n\pi}{L}\right)^2$$

Completeness of eigenfunctions:

$$\sigma(x)\sum_n\frac{f_n(x)f_n(x')}{N_n}=\delta(x-x')$$

In this example: $\frac{2}{L}\sum_n\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi x'}{L}\right)=\delta(x-x')$

Green's function:

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)G_\lambda(x,x')=\delta(x-x')$$

Green's function for the example:

$$G(x,x')=\sum_n\frac{f_n(x)f_n(x')/N_n}{\lambda_n-\lambda}=\frac{2}{L}\sum_n\frac{\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi x'}{L}\right)}{\left(\frac{n\pi}{L}\right)^2-1}$$

Using Green's function to solve inhomogenous equation :

$$\left(-\frac{d^2}{dx^2}-1\right)\phi(x)=F_0\sin\left(\frac{\pi x}{L}\right)$$

$$\phi(x)=\phi_0(x)+\int_0^L G(x,x')F_0\sin\left(\frac{\pi x'}{L}\right)dx'$$

$$=\phi_0(x)+\frac{2}{L}\sum_n\left[\frac{\sin\left(\frac{n\pi x}{L}\right)}{\left(\frac{n\pi}{L}\right)^2-1}\int_0^L\sin\left(\frac{n\pi x'}{L}\right)F_0\sin\left(\frac{\pi x'}{L}\right)dx'\right]$$

$$=\phi_0(x)+\frac{F_0}{\left(\frac{\pi}{L}\right)^2-1}\sin\left(\frac{\pi x}{L}\right)$$

Alternate Green's function method :

$$G(x,x')=\frac{1}{W}g_a(x_<)g_B(x_>)$$

$$\left(-\frac{d^2}{dx^2}-1\right)g_i(x)=0 \qquad \Rightarrow g_a(x)=\sin(x);\quad g_b(x)=\sin(L-x);$$

$$W=g_b(x)\frac{dg_a(x)}{dx}-g_a(x)\frac{dg_b(x)}{dx}=\sin(L-x)\cos(x)+\sin(x)\cos(L-x)$$

$$=\ \sin(L)$$

$$\phi(x)=\phi_0(x)+\frac{\sin(L-x)}{\sin(L)}\int_0^x\sin(x')F_0\sin\left(\frac{\pi x'}{L}\right)dx'$$

$$+\frac{\sin(x)}{\sin(L)}\int_x^L\sin(L-x')F_0\sin\left(\frac{\pi x'}{L}\right)dx'$$

$$\phi(x)=\phi_0(x)+\frac{F_0}{\left(\frac{\pi}{L}\right)^2-1}\sin\left(\frac{\pi x}{L}\right)$$

General method of constructing Green's functions using homogeneous solution

Green's function :

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)G_\lambda(x,x')=\delta(x-x')$$

Two homogeneous solutions

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)g_i(x)=0 \qquad \text{for} \quad i=a,b$$

Let

$$G_\lambda(x,x')=\frac{1}{W}g_a(x_<)g_b(x_>)$$

For $\epsilon \to 0$:

$$\int_{x'-\epsilon}^{x'+\epsilon} dx \left(-\frac{d}{dx}\tau(x)\frac{d}{dx} + v(x) - \lambda\sigma(x) \right) G_\lambda(x,x') = \int_{x'-\epsilon}^{x'+\epsilon} dx \delta(x-x')$$

$$\int_{x'-\epsilon}^{x'+\epsilon} dx \left(-\frac{d}{dx}\tau(x)\frac{d}{dx} \right) \frac{1}{W} g_a(x_<) g_b(x_>) = 1$$

$$-\frac{\tau(x)}{W}\left(\frac{d}{dx} g_a(x_<) g_b(x_>) \right)\Bigg]_{x'-\epsilon}^{x'+\epsilon} = \frac{\tau(x')}{W}\left(g_a(x')\frac{d}{dx} g_b(x') - g_b(x')\frac{d}{dx} g_a(x') \right)$$

$$\Rightarrow W = \tau(x')\left(g_a(x')\frac{d}{dx} g_b(x') - g_b(x')\frac{d}{dx} g_a(x') \right)$$

Note -- W (Wronskian) is constant, since $\frac{dW}{dx'} = 0.$

$\Rightarrow$ Useful Green's function construction in one dimension:

$$G_\lambda(x,x') = \frac{1}{W} g_a(x_<) g_b(x_>)$$

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx} + v(x) - \lambda\sigma(x) \right)\varphi(x) = F(x)$$

Green's function solution:

$$\varphi_\lambda(x) = \varphi_{\lambda 0}(x) + \int_{x_l}^{x_u} G_\lambda(x,x')F(x')dx'$$

$$= \varphi_{\lambda 0}(x) + \frac{g_b(x)}{W}\int_{x_l}^{x} g_a(x')F(x')dx' + \frac{g_a(x)}{W}\int_{x}^{x_u} g_b(x')F(x')dx'$$