

**PHY 711 Classical Mechanics and
Mathematical Methods**
9-9:50 AM MWF Olin 107

Plan for Lecture 24:

Read Chapter 7 & Appendices A-D

Generalization of the one dimensional wave equation →
various mathematical problems and techniques including:

1. Laplace transforms
2. Complex variables
3. Contour integrals

10/27/2017

PHY 711 Fall 2017 – Lecture 24

1

17 Mon, 10/09/2017	Chap. 5	Right body motion	#10	10/16/2017
18 Wed, 10/11/2017	Chap. 5	Right body motion		
Fri, 10/13/2017		Fall break – No class		
19 Mon, 10/16/2017		Discuss exam questions and topics for presentations	Topic	10/18/2017
20 Wed, 10/18/2017	Chap. 7	Wave equation in one dimension	#11	10/20/2017
21 Fri, 10/20/2017	Chap. 7	Solutions of Sturm-Liouville equations	#12	10/27/2017
22 Mon, 10/23/2017	Chap. 7	Solutions of Sturm-Liouville equations		
23 Wed, 10/25/2017	Chap. 7	Solutions of Sturm-Liouville equations		
24 Fri, 10/27/2017	App. A	Laplace transforms and contour integrals	#13	11/01/2017
25 Mon, 10/30/2017				
26 Wed, 11/01/2017				
27 Fri, 11/03/2017				
28 Mon, 11/06/2017				
29 Wed, 11/08/2017				
30 Fri, 11/10/2017				
31 Mon, 11/13/2017				
32 Wed, 11/15/2017				
33 Fri, 11/17/2017				
34 Mon, 11/20/2017				

10/27/2017

PHY 711 Fall 2017 – Lecture 24

2

Laplace transforms

Laplace transforms can be used to solve initial value problems. The Laplace transform of a function $\phi(x)$ is defined as

$$\mathcal{L}_{\phi}(p) \equiv \int_0^{\infty} e^{-px} \phi(x) dx. \quad (24)$$

Assuming that $\phi(x)$ is well-behaved in the interval $0 \leq x \leq \infty$, the following properties are useful:

$$\mathcal{L}_{d\phi/dx}(p) = -\phi(0) + p\mathcal{L}_{\phi}(p), \quad (25)$$

and

$$\mathcal{L}_{d^2\phi/dx^2}(p) = -\frac{d\phi(0)}{dx} - p\phi(0) + p^2\mathcal{L}_{\phi}(p). \quad (26)$$

10/27/2017

PHY 711 Fall 2017 – Lecture 24

3

These identities allow us to turn a differential equation for $\phi(x)$ into an algebraic equation for $\mathcal{L}_\phi(p)$. We then need to perform an inverse Laplace transform to find $\phi(x)$. For illustration, we will consider a simple example with $\tau(x) = 1$, $\sigma(x) = 1$, $\lambda = 0$. The differential equation then becomes

$$-\frac{d^2\phi(x)}{dx^2} = F(x), \quad (27)$$

where we will take the initial conditions to be $\phi(0) = 0$ and $d\phi(0)/dx = 0$. For our example, we will also take $F(x) = F_0 e^{-\gamma x}$. Multiplying both sides of the equation by e^{-px} and integrating $0 \leq x \leq \infty$, we find

$$\mathcal{L}_\phi(p) = -\frac{F_0}{p^2(\gamma + p)} \quad (28)$$

10/27/2017

PHY 711 Fall 2017 -- Lecture 24

4

In general the inverse Laplace transform involves performing a contour integral, but we can use the following simple relations

$$\mathcal{L}_1 = \int_0^\infty e^{-px} dx = \frac{1}{p}. \quad (29)$$

$$\mathcal{L}_x = \int_0^\infty x e^{-px} dx = \frac{1}{p^2}. \quad (30)$$

$$\mathcal{L}_{e^{-\gamma x}} = \int_0^\infty e^{-\gamma x} e^{-px} dx = \frac{1}{p + \gamma}. \quad (31)$$

Noting that

$$-\frac{F_0}{p^2(\gamma + p)} = -\frac{F_0}{\gamma^2} \left(\frac{1}{\gamma + p} - \frac{1}{p} + \frac{\gamma}{p^2} \right), \quad (32)$$

we see that the inverse Laplace transform gives us

$$\phi(x) = \frac{F_0}{\gamma^2} (1 - e^{-\gamma x} - \gamma x). \quad (33)$$

We can check that this is a solution to the differential equation

$$-\frac{d^2\phi}{dx^2} = F_0 e^{-\gamma x} \quad \text{for} \quad \phi(0) = 0 \quad \text{and} \quad \frac{d\phi}{dx}(0) = 0$$

10/27/2017

PHY 711 Fall 2017 -- Lecture 24

5

Using Laplace transforms to solve equation:

$$\left(-\frac{d^2}{dx^2} - 1\right)\phi(x) = F_0 \sin\left(\frac{\pi x}{L}\right) \quad \text{with} \quad \phi(0) = 0, \quad \frac{d\phi(0)}{dx} = 0$$

$$\mathcal{L}_\phi(p) = -\left(\frac{\pi}{L}\right) \frac{F_0}{\left(p^2 + 1\right)\left(p^2 + \left(\frac{\pi}{L}\right)^2\right)}$$

$$= -F_0 \left(\frac{\pi/L}{\left(\pi/L\right)^2 - 1} \right) \left(\frac{1}{p^2 + 1} - \frac{1}{p^2 + \left(\frac{\pi}{L}\right)^2} \right)$$

$$\text{Note that: } \int_0^\infty \sin(at) e^{-pt} dt = \frac{a}{a^2 + p^2}$$

$$\Rightarrow \phi(x) = \frac{F_0}{\left(\pi/L\right)^2 - 1} \left(\sin\left(\frac{\pi x}{L}\right) - \frac{\pi}{L} \sin(x) \right)$$

10/27/2017

PHY 711 Fall 2017 -- Lecture 24

6

Inverse Laplace transform :

$$\mathcal{L}_p(p) = \int_0^{\infty} e^{-pt} \phi(t) dt$$

$$\phi(t) = \frac{1}{2\pi i} \int_{\lambda-i\infty}^{\lambda+i\infty} e^{pt} \mathcal{L}_p(p) dp$$

Check: $\frac{1}{2\pi i} \int_{\lambda-i\infty}^{\lambda+i\infty} e^{pt} \mathcal{L}_p(p) dp = \frac{1}{2\pi i} \int_{\lambda-i\infty}^{\lambda+i\infty} e^{pt} dp \int_0^{\infty} e^{-pu} \phi(u) du$

$$\begin{aligned} \frac{1}{2\pi i} \int_0^{\infty} \phi(u) du \int_{\lambda-i\infty}^{\lambda+i\infty} e^{p(t-u)} dp &= \frac{1}{2\pi i} \int_0^{\infty} \phi(u) du \int_{-\infty}^{\infty} e^{\lambda(t-u)} e^{is(t-u)} ds \\ &= \frac{1}{2\pi i} \int_0^{\infty} \phi(u) du \left(e^{\lambda(t-u)} 2\pi i \delta(t-u) \right) \\ &= \begin{cases} \phi(t) & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

10/27/2017

PHY 711 Fall 2017 -- Lecture 24

7

Complex numbers

$$i \equiv \sqrt{-1} \quad i^2 = -1$$

Define $z = x + iy$

$$|z|^2 = zz^* = (x + iy)(x - iy) = x^2 + y^2$$

Polar representation

$$z = \rho(\cos \phi + i \sin \phi) = \rho e^{i\phi}$$

Functions of complex variables

$$f(z) = \Re(f(z)) + i\Im(f(z)) \equiv u(x, y) + iv(x, y)$$

Derivatives: Cauchy-Riemann equations

$$\frac{\partial f(z)}{\partial x} = \frac{\partial u(z)}{\partial x} + i \frac{\partial v(z)}{\partial x} \quad \frac{\partial f(z)}{\partial y} = \frac{\partial u(z)}{\partial y} + i \frac{\partial v(z)}{\partial y} = \frac{\partial v(z)}{\partial y} - i \frac{\partial u(z)}{\partial y}$$

Argue that $\frac{df}{dz} = \frac{\partial f(z)}{\partial x} = \frac{\partial f(z)}{\partial y} \Rightarrow \frac{\partial u(z)}{\partial x} = \frac{\partial v(z)}{\partial y} \quad \text{and} \quad \frac{\partial v(z)}{\partial x} = -\frac{\partial u(z)}{\partial y}$

10/27/2017

PHY 711 Fall 2017 -- Lecture 24

8

Analytic function

$f(z)$ is analytic if it is:

- continuous
- single valued
- its first derivative satisfies Cauchy-Riemann conditions

Which of the following functions are analytic?

$$f(z) = e^z$$

$$f(z) = z^n$$

$$f(z) = \ln z$$

$$f(z) = z^\alpha$$

10/27/2017

PHY 711 Fall 2017 -- Lecture 24

9

Some details

$$e^z = e^{x+iy} = e^x \cos(y) + ie^x \sin(y)$$

$$\frac{\partial u}{\partial x} = e^x \cos(y) = \frac{\partial v}{\partial y} \quad \frac{\partial v}{\partial x} = e^x \sin(y) = -\frac{\partial u}{\partial y}$$

$$z^2 = (x + iy)^2 = (x^2 - y^2) + 2ixy$$

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y} \quad \frac{\partial v}{\partial x} = 2y = -\frac{\partial u}{\partial y}$$

10/27/2017

PHY 711 Fall 2017 -- Lecture 24

10

PHY 711 – Contour Integration

These notes summarize some basic properties of complex functions and their integrals. An *analytic* function $f(z)$ in a certain region of the complex plane z is one which takes a single (non-infinite) value and is differentiable within that region. Cauchy's theorem states that a closed contour integral of the function within that region has the value

$$\oint_C f(z) dz = 0. \quad (1)$$

As an example, functions composed of integer powers of z –

$$f(z) = z^n, \quad \text{for } n = 0, 1, \pm 2, \pm 3, \dots \quad (2)$$

fall in this category. Notice that non-integer powers are generally not analytic and that $n = -1$ is also special. In fact, we can show that

$$\oint_C \frac{dz}{z} = 2\pi i. \quad (3)$$

10/27/2017

PHY 711 Fall 2017 -- Lecture 24

11

$$\oint_C \frac{dz}{z} = 2\pi i. \quad (3)$$

This result follows from the fact that we can deform the contour to a unit circle about the origin so that $z = e^{i\theta}$. Then

$$\oint_C \frac{dz}{z} = \int_0^{2\pi} \frac{e^{i\theta} i d\theta}{e^{i\theta}} = 2\pi i. \quad (4)$$

10/27/2017

PHY 711 Fall 2017 -- Lecture 24

12

Another result of this analysis is the Residue Theorem which states that if the complex function $g(z)$ has poles at a finite number of points z_p within a region C but is otherwise analytic, the contour integral can be evaluated according to

$$\oint_C g(z) dz = 2\pi i \sum_p \text{Res}(g_p), \quad (6)$$

where the residue is given by

$$\text{Res}(g_p) \equiv \lim_{z \rightarrow z_p} \left\{ \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} ((z - z_p)^m g(z)) \right\}, \quad (7)$$

where m denotes the order of the pole.

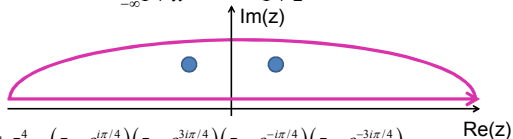
10/27/2017

PHY 711 Fall 2017 – Lecture 24

13

Example:

$$\int_{-\infty}^{\infty} \frac{x^2}{1+x^4} dx = \oint \frac{z^2}{1+z^4} dz$$



$$1+z^4 = (z - e^{i\pi/4})(z - e^{3i\pi/4})(z - e^{-i\pi/4})(z - e^{-3i\pi/4})$$

$$\oint \frac{z^2}{1+z^4} dz = 2\pi i \left(\text{Res}(z_p = e^{i\pi/4}) + \text{Res}(z_p = e^{3i\pi/4}) \right)$$

$$\text{Res}(z_p = e^{i\pi/4}) = \frac{e^{i\pi/4}}{4i} \quad \text{Res}(z_p = e^{3i\pi/4}) = -\frac{e^{3i\pi/4}}{4i}$$

$$\oint \frac{z^2}{1+z^4} dz = 2\pi i \left(\frac{e^{i\pi/4}}{4i} - \frac{e^{3i\pi/4}}{4i} \right) = \frac{\pi}{\sqrt{2}}$$

10/27/2017

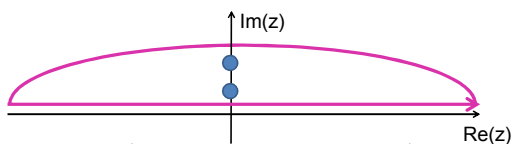
PHY 711 Fall 2017 – Lecture 24

14

Another example: $I = \int_0^{\infty} \frac{\cos(ax)}{4x^4 + 5x^2 + 1} dx.$

$$\int_0^{\infty} \frac{\cos(ax)}{4x^4 + 5x^2 + 1} dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{e^{iax}}{4x^4 + 5x^2 + 1} dx = \frac{1}{2} \oint \frac{e^{iaz}}{4z^4 + 5z^2 + 1} dz$$

$$4z^4 + 5z^2 + 1 = 4(z-i)\left(z-\frac{i}{2}\right)(z+i)\left(z+\frac{i}{2}\right)$$



$$I = 2\pi i \left(\text{Res}(z_p = i) + \text{Res}(z_p = \frac{i}{2}) \right)$$

$$\begin{aligned}
 \int_0^{\infty} \frac{\cos(ax)}{4x^4 + 5x^2 + 1} dx &= \frac{1}{2} \oint \frac{e^{iaz}}{4z^4 + 5z^2 + 1} dz \\
 &= 2\pi i \left(\text{Res}\left(z_p = i\right) + \text{Res}\left(z_p = \frac{i}{2}\right) \right) \\
 &= \frac{\pi}{6} \left(-e^{-a} + 2e^{-a/2} \right)
 \end{aligned}$$

10/27/2017

PHY 711 Fall 2017 -- Lecture 24

16

Cauchy integral theorem for analytic function $f(z)$:

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(z')}{z' - z} dz'.$$

10/27/2017

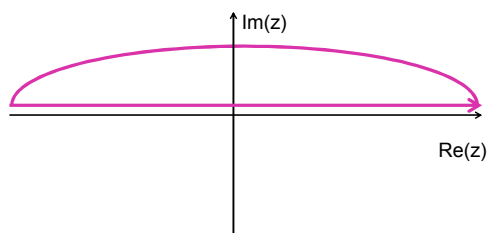
PHY 711 Fall 2017 -- Lecture 24

17

Example

Suppose $f(|z| \rightarrow \infty) = 0$ and for $z = x$:

$$f(x) = a(x) + ib(x)$$



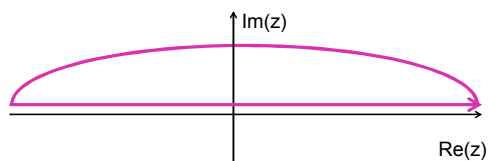
10/27/2017

PHY 711 Fall 2017 -- Lecture 24

18

Example -- continued

$$f(z) = \frac{1}{2\pi i} \oint \frac{f(z')}{z' - z} dz' \quad \text{where } f(x) = a(x) + ib(x)$$



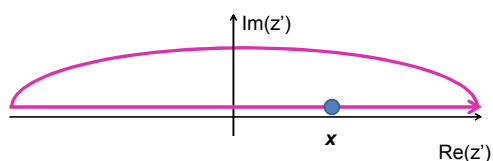
$$a(x) + ib(x) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{a(x') + ib(x')}{x' - x} dx'$$

10/27/2017

PHY 711 Fall 2017 -- Lecture 24

19

Example -- continued



$$\begin{aligned} \int_{-\infty}^{\infty} \frac{f(x')}{x' - x} dx' &= \int_{-\infty}^{x-\epsilon} \frac{f(x')}{x' - x} dx' + \int_{x+\epsilon}^{\infty} \frac{f(x')}{x' - x} dx' + \int_{x-\epsilon}^{x+\epsilon} \frac{f(x')}{x' - x} dx' \\ &= P \int_{-\infty}^{\infty} \frac{f(x')}{x' - x} dx' + i\pi f(x) \end{aligned}$$

10/27/2017

PHY 711 Fall 2017 -- Lecture 24

20

Example -- continued

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{f(x')}{x' - x} dx' &= \int_{-\infty}^{x-\epsilon} \frac{f(x')}{x' - x} dx' + \int_{x+\epsilon}^{\infty} \frac{f(x')}{x' - x} dx' + \int_{x-\epsilon}^{x+\epsilon} \frac{f(x')}{x' - x} dx' \\ &= P \int_{-\infty}^{\infty} \frac{f(x')}{x' - x} dx' + i\pi f(x) \end{aligned}$$

$$a(x) + ib(x) = \frac{P}{2\pi i} \int_{-\infty}^{\infty} \frac{a(x') + ib(x')}{x' - x} dx' + \frac{\pi i}{2\pi i} (a(x) + ib(x))$$

$$\Rightarrow a(x) = \frac{P}{\pi} \int_{-\infty}^{\infty} \frac{b(x')}{x' - x} dx' \quad b(x) = -\frac{P}{\pi} \int_{-\infty}^{\infty} \frac{a(x')}{x' - x} dx'$$

Kramers-Kronig relationships

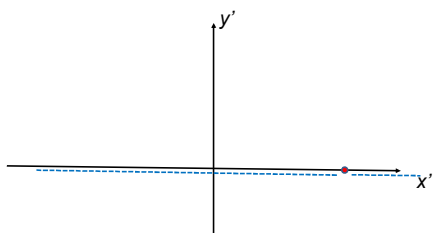
10/27/2017

PHY 711 Fall 2017 -- Lecture 24

21

Comment on evaluating principal parts integrals

$$a(x) = \frac{P}{\pi} \int_{-\infty}^{\infty} \frac{b(x')}{x' - x} dx' = \lim_{\epsilon \rightarrow 0} \left(\frac{1}{\pi} \int_{-\infty}^{x-\epsilon} \frac{b(x')}{x' - x} dx' + \frac{1}{\pi} \int_{x+\epsilon}^{\infty} \frac{b(x')}{x' - x} dx' \right)$$



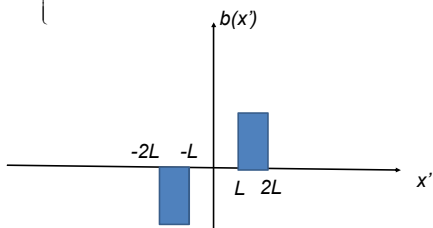
10/27/2017

PHY 711 Fall 2017 – Lecture 24

22

Example:

$$b(x') = \begin{cases} 0 & \text{for } x' < -2L, \quad -L < x' < L, \quad x' > 2L \\ B_0 & \text{for } L < x' < 2L \\ -B_0 & \text{for } -2L < x' < -L \end{cases}$$



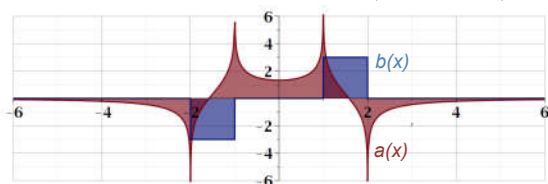
10/27/2017

PHY 711 Fall 2017 – Lecture 24

23

$$a(x) = \frac{P}{\pi} \int_{-\infty}^{\infty} \frac{b(x')}{x' - x} dx' = \lim_{\epsilon \rightarrow 0} \left(\frac{1}{\pi} \int_{-\infty}^{x-\epsilon} \frac{b(x')}{x' - x} dx' + \frac{1}{\pi} \int_{x+\epsilon}^{\infty} \frac{b(x')}{x' - x} dx' \right)$$

For our example:
$$a(x) = \frac{B_0}{\pi} \ln \left(\left| \frac{4L^2 - x^2}{L^2 - x^2} \right| \right)$$



10/27/2017

PHY 711 Fall 2017 – Lecture 24

24
