PHY 711 Classical Mechanics and Mathematical Methods
9-9:50 AM MWF Olin 107
Plan for Lecture 26:

Motions of elastic membranes (Chap. 8)

1. Review of standing waves on a string
2. Standing waves on a two dimensional membrane.
3. Boundary value problems

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Events

Meet the Speaker: Nov 1 at Noon
WFU Physics Meet the Speaker Event SPEAKER: Dr. Yves Chabal Department of Materials Science and Engineering, University of Texas at Dallas, Dallas, TX TIME: Wed. Nov. 1, 2017, 12:00 – 1:00 ...

Colloquium: Nov. 1, 2017 at 4 PM
WFU Physics Colloquium TITLE: "The fascinating world of Metal Organic Framework materials" SPEAKER: Yves Chabal Department of Materials Science and Engineering, University of Texas at Dallas, Dallas, TX TIME: Wed. ...

19	Mon, 10/16/2017		Discuss exam questions and topics for presentations	Topic	10/18/2017
20	Wed, 10/18/2017	Chap. 7	Wave equation in one dimension	#11	10/20/2017
21	Fri, 10/20/2017	Chap. 7	Solutions of Sturm-Liouville equations	#12	10/27/2017
22	Mon, 10/23/2017	Chap. 7	Solutions of Sturm-Liouville equations		
23	Wed, 10/25/2017	Chap. 7	Solutions of Sturm-Liouville equations		
24	Fri, 10/27/2017	App. A	Laplace transforms and contour integrals	#13	11/01/2017
25	Mon, 10/30/2017	App. A	Contour integrals		
26	Wed, 11/01/2017	Chap. 8	Mechanics of Elastic Membranes	#14	11/06/2017
27	Fri, 11/03/2017				
28	Mon, 11/06/2017				
29	Wed, 11/08/2017				
30	Fri, 11/10/2017				
31	Mon, 11/13/2017				
32	Wed, 11/15/2017				
33	Fri, 11/17/2017				
34	Mon, 11/20/2017				
	Wed, 11/22/2017		*Thanksgiving Holiday -- No class*		
	Fri, 11/24/2017		*Thanksgiving Holiday -- No class*		
35	Mon, 11/27/2017				

Elastic media in two or more dimensions --

Review of wave equation in one-dimension – here $\mu(x,t)$ can describe either a longitudinal or transverse wave.

Traveling wave solutions --

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0$$

Note that for any function $f(q)$ or $g(q)$:

$$\mu(x,t) = f(x-ct) + g(x+ct)$$

satisfies the wave equation.

Initial value problem: $\mu(x,0) = \phi(x)$ and $\frac{\partial \mu}{\partial t}(x,0) = \psi(x)$

then: $\mu(x,0) = \phi(x) = f(x) + g(x)$

$$\frac{\partial \mu}{\partial t}(x,0) = \psi(x) = -c\left(\frac{df(x)}{dx} - \frac{dg(x)}{dx}\right)$$

$$\Rightarrow f(x) - g(x) = -\frac{1}{c}\int^{x} \psi(x')dx'$$

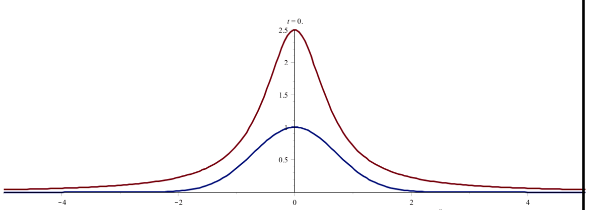
For each x, find $f(x)$ and $g(x)$:

$$f(x) = \frac{1}{2}\left(\phi(x) - \frac{1}{c}\int^{x} \psi(x')dx'\right)$$

$$g(x) = \frac{1}{2}\left(\phi(x) + \frac{1}{c}\int^{x} \psi(x')dx'\right)$$

$$\Rightarrow \mu(x,t) = \frac{1}{2}\left(\phi(x-ct) + \phi(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(x')dx'$$

Example with $\psi(x) = 0$ and $\phi(x) = \frac{1}{x^2 + 0.4}$



Example with $\psi(x) = 0$ and $\phi(x) = e^{-x^2}$

Standing wave solutions of wave equation:

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0$$

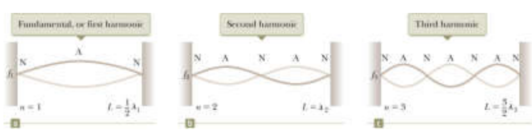
with $\mu(0,t) = \mu(L,t) = 0$.

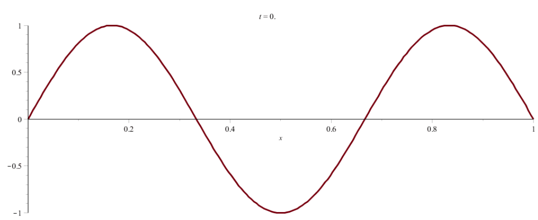
Assume: $\mu(x,t) = \Re\left(e^{-i\omega t}\rho(x)\right)$

where $\dfrac{d^2\rho(x)}{dx^2} + k^2\rho(x) = 0 \qquad k = \dfrac{\omega}{c}$

$$\rho_\nu(x) = A\sin\left(\frac{\nu\pi x}{L}\right)$$

$$k_\nu = \frac{\nu\pi}{L} \qquad \omega_\nu = ck_\nu$$





Wave motion on a two-dimensional surface – elastic membrane (transverse wave; linear regime).

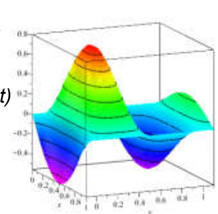
Two-dimensional wave equation:

$$\frac{\partial^2 u}{\partial t^2} - c^2\nabla^2 u = 0 \qquad \text{where } c^2 = \frac{\tau}{\sigma}$$

Standing wave solutions:

$$u(x,y,t) = \Re\left(e^{-i\omega t}\rho(x,y)\right)$$

$$\left(\nabla^2 + k^2\right)\rho(x,y) = 0 \qquad \text{where } k = \frac{\omega}{c}$$



Lagrangian density: $\mathcal{L}\left(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial t}; x, y, t\right)$

$$L = \int \mathcal{L}\left(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial t}; x, y, t\right) dxdy$$

Hamilton's principle:

$$\delta \int_{t_1}^{t_2} L dt = 0$$

$$\frac{\partial \mathcal{L}}{\partial u} - \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial(\partial u/\partial t)} - \frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial(\partial u/\partial x)} - \frac{\partial}{\partial y}\frac{\partial \mathcal{L}}{\partial(\partial u/\partial y)} = 0$$

Lagrangian density for elastic membrane with constant σ and τ :

$$\mathcal{L}\left(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial t}; x, y, t\right) = \frac{1}{2}\sigma\left(\frac{\partial u}{\partial t}\right)^2 - \frac{1}{2}\tau(\nabla u)^2$$

$$\frac{\partial \mathcal{L}}{\partial u} - \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial(\partial u/\partial t)} - \frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial(\partial u/\partial x)} - \frac{\partial}{\partial y}\frac{\partial \mathcal{L}}{\partial(\partial u/\partial y)} = 0$$

$$\frac{\partial^2 u}{\partial t^2} - c^2\nabla^2 u = 0 \qquad \text{where } c^2 = \frac{\tau}{\sigma}$$

Two-dimensional wave equation:

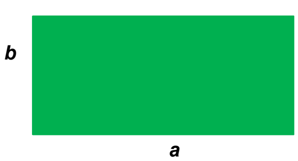
$$\frac{\partial^2 u}{\partial t^2} - c^2\nabla^2 u = 0 \qquad \text{where } c^2 = \frac{\tau}{\sigma}$$

Standing wave solutions:

$$u(x, y, t) = \Re\left(e^{-i\omega t}\rho(x, y)\right)$$

$$\left(\nabla^2 + k^2\right)\rho(x, y) = 0 \qquad \text{where } k = \frac{\omega}{c}$$

Consider a rectangular boundary:



Clamped boundary conditions:

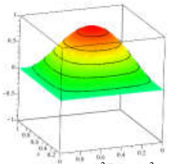
$$\rho(0, y) = \rho(a, y) = \rho(x, 0) = \rho(x, b) = 0$$

$$\Rightarrow \rho_{mn}(x, y) = A \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

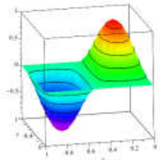
$$k_{mn}{}^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \qquad \omega_{mn} = ck_{mn}$$

$$\left(\nabla^2 + k^2\right)\rho(x, y) = 0$$

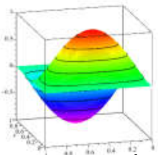
$$\text{where } k = \frac{\omega}{c}$$



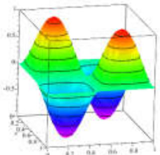
$k_{11}^2 = \left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2$



$k_{12}^2 = \left(\frac{\pi}{a}\right)^2 + \left(\frac{2\pi}{b}\right)^2$



$k_{21}^2 = \left(\frac{2\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2$



$k_{22}^2 = \left(\frac{2\pi}{a}\right)^2 + \left(\frac{2\pi}{b}\right)^2$

More general boundary conditions:

$\tau\nabla u\big|_b = \kappa u\big|_b$ represents bounded side constrained with spring

$\tau\nabla u\big|_b = 0$ represents "free" side

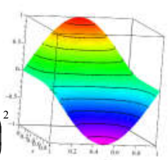
Mixed boundary conditions:

$$\rho(x,0) = \rho(x,b) = \frac{\partial\rho(0,y)}{\partial x} = \frac{\partial\rho(a,y)}{\partial x} = 0$$

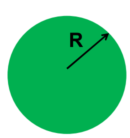
$$\Rightarrow \rho_{mn}(x,y) = A\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

$$k_{mn}^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \qquad \omega_{mn} = ck_{mn}$$

$k_{11}^2 = \left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2$



Consider a circular boundary:



Clamped boundary conditions for $\rho(r,\varphi)$:

$\rho(R,\varphi) = 0$

$$\left(\nabla^2 + k^2\right)\rho(r,\varphi) = 0 \qquad \text{where } k = \frac{\omega}{c}$$

In cylindrical coordinate system

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}$$

Assume: $\rho(r,\varphi) = f(r)\Phi(\varphi)$

Let: $\Phi(\varphi) = e^{im\varphi}$

Note: $\Phi(\varphi) = \Phi(\varphi + 2\pi)$

$\Rightarrow m = \text{integer}$

Consider circular boundary -- continued

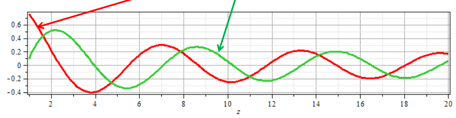
Differential equation for radial function:

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{m^2}{r^2}+k^2\right)f(r)=0$$

$\Rightarrow$ Bessel equation of integer order with transcendential solutions

Cylindrical Bessel function $J_m(z)$

Cylindrical Neumann function $N_m(z)$ also called $Y_m(z)$



Some properties of Bessel functions

Asending series: $J_m(z)=\left(\frac{z}{2}\right)^m\sum_{j=0}^{\infty}\frac{(-1)^j}{j!(j+m)!}\left(\frac{z}{2}\right)^{2j}$

Recursion relations: $J_{m-1}(z)+J_{m+1}(z)=\frac{2m}{z}J_m(z)$

$$J_{m-1}(z)-J_{m+1}(z)=2\frac{dJ_m(z)}{dz}$$

Asymptotic form: $J_m(z)\xrightarrow[z>>1]{}\sqrt{\frac{2}{\pi z}}\cos\left(z-\frac{m\pi}{2}-\frac{\pi}{4}\right)$

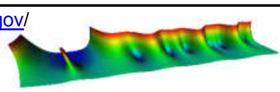
Zeros of Bessel functions $J_m(z_{mn})=0$

$m=0$: $z_{0n}=2.406,\ 5.520,\ 8.654,...$

$m=1$: $z_{1n}=3.832,\ 7.016,\ 10.173,...$

$m=2$: $z_{2n}=5.136,\ 8.417,\ 11.620,...$

http://dlmf.nist.gov/



NIST Digital Library of Mathematical Functions

Project News

2014-08-29 DLMF Update; Version 1.0.9
2014-04-25 DLMF Update; Version 1.0.8; errata & improved MathML
2014-03-21 DLMF Update; Version 1.0.7; New Features improve Math & 3D Graphics
2013-08-16 Bille C. Carlson, DLMF Author, dies at age 89
More news

Foreword
Preface
Mathematical Introduction
1 Algebraic and Analytic Methods
2 Asymptotic Approximations
3 Numerical Methods
4 Elementary Functions
5 Gamma Function
6 Exponential, Logarithmic, Sine, and Cosine Integrals
7 Error Functions, Dawson's and Fresnel Integrals
8 Incomplete Gamma and Related Functions
9 Airy and Related Functions
10 Bessel Functions
11 Struve and Related Functions
12 Parabolic Cylinder Functions
19 Elliptic Integrals
20 Theta Functions
21 Multidimensional Theta Functions
22 Jacobian Elliptic Functions
23 Weierstrass Elliptic and Modular Functions
24 Bernoulli and Euler Polynomials
25 Zeta and Related Functions
26 Combinatorial Analysis
27 Functions of Number Theory
28 Mathieu Functions and Hill's Equation
29 Lamé Functions
30 Spheroidal Wave Functions
31 Heun Functions
32 Painlevé Transcendents
33 Coulomb Functions

Series expansions of Bessel and Neumann functions

$$J_\nu(z) = \left(\tfrac{1}{2}z\right)^\nu \sum_{k=0}^{\infty} (-1)^k \frac{\left(\tfrac{1}{4}z^2\right)^k}{k!\,\Gamma(\nu+k+1)}.$$

$$Y_n(z) = -\frac{\left(\tfrac{1}{2}z\right)^{-n}}{\pi}\sum_{k=0}^{n-1}\frac{(n-k-1)!}{k!}\left(\tfrac{1}{4}z^2\right)^k + \frac{2}{\pi}\ln\left(\tfrac{1}{2}z\right)J_n(z)$$
$$-\frac{\left(\tfrac{1}{2}z\right)^n}{\pi}\sum_{k=0}^{\infty}\left(\psi(k+1)+\psi(n+k+1)\right)\frac{\left(-\tfrac{1}{4}z^2\right)^k}{k!(n+k)!},$$

Some properties of Bessel functions -- continued

Note : It is possible to prove the following identity for the functions $J_m\left(\frac{z_{mn}}{R}r\right)$:

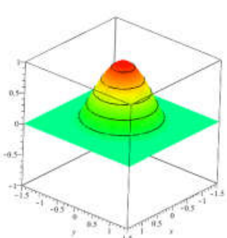
$$\int_0^R J_m\left(\frac{z_{mn}}{R}r\right)J_m\left(\frac{z_{mn'}}{R}r\right)rdr = \frac{R^2}{2}\left(J_{m+1}(z_{mn})\right)^2\delta_{nn'}$$

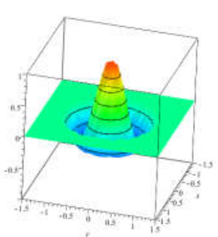
Returning to differential equation for radial function :

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{m^2}{r^2}+k^2\right)f(r)=0$$

$$\Rightarrow\ f_{mn}(r) = AJ_m\left(\frac{z_{mn}}{R}r\right); \qquad k_{mn} = \frac{z_{mn}}{R}$$

$$\rho_{01}(r,\varphi) = f_{01}(r) = AJ_0\left(\frac{z_{01}}{R}r\right) \qquad \rho_{02}(r,\varphi) = f_{02}(r) = AJ_0\left(\frac{z_{02}}{R}r\right)$$

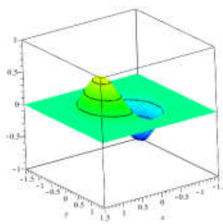


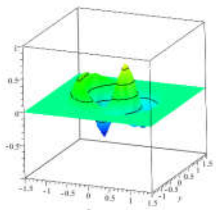


$$k_{01} = \frac{2.406}{R} \qquad k_{02} = \frac{5.520}{R}$$

$$\rho_{11}(r,\varphi) = f_{11}(r)\cos(\varphi) = AJ_1\left(\frac{z_{11}}{R}r\right)\cos(\varphi)$$

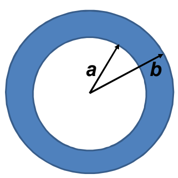
$$\rho_{12}(r,\varphi) = f_{12}(r)\cos(\varphi) = AJ_1\left(\frac{z_{12}}{R}r\right)\cos(\varphi)$$





$$k_{11} = \frac{3.832}{R} \qquad k_{12} = \frac{7.016}{R}$$

More complicated geometry – annular membrane



In cylindrical coordinate system

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}$$

Assume : $\rho(r,\varphi) = f(r)\Phi(\varphi)$

Let : $\Phi(\varphi) = e^{im\varphi}$

Note : $\Phi(\varphi) = \Phi(\varphi + 2\pi)$

$\Rightarrow m = \text{integer}$

Consider circular boundary -- continued

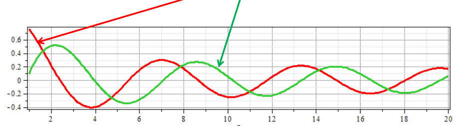
Differential equation for radial function :

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m^2}{r^2} + k^2\right)f(r) = 0$$

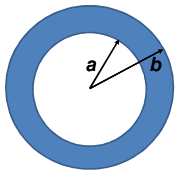
$\Rightarrow$ Bessel equation of integer order with transcendential solutions

Cylindrical Bessel function $J_m(z)$

Cylindrical Neumann function $N_m(z)$



Normal modes of an annular membrane -- continued



Differential equation for radial function:

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{m^2}{r^2}+k^2\right)f(r)=0$$

General form of radial function: $f(r)=AJ_m(kr)+BN_m(kr)$

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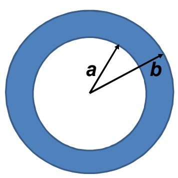
Normal modes of an annular membrane -- continued



Boundary conditions:
$f(a)=0 \qquad f(b)=0$

$$AJ_m(ka)+BN_m(ka)=0$$
$$AJ_m(kb)+BN_m(kb)=0$$

$\Rightarrow$ 2 equations and 2 unknowns -- k and $\frac{B}{A}$

$$\frac{B}{A}=\frac{-J_m(ka)}{N_m(ka)}=\frac{-J_m(kb)}{N_m(kb)} \quad \text{(transcendental equation for } k\text{)}$$

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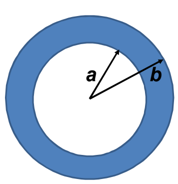
Normal modes of an annular membrane -- continued



Boundary conditions:
$f(a)=0 \qquad f(b)=0$

$$\frac{B}{A}=\frac{-J_m(ka)}{N_m(ka)}=\frac{-J_m(kb)}{N_m(kb)} \quad \text{-- in terms of solution } k_{mn}:$$

$$f(r)=A\left(J_m(k_{mn}r)-\frac{J_m(k_{mn}a)}{N_m(k_{mn}a)}N_m(k_{mn}r)\right)$$

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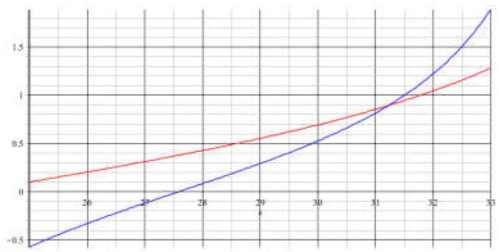
Analysis for *m=0* and a=0.1, b=0.2:

$$> plot\left(\left\{\frac{-\text{BesselJ}(0, 0.1\cdot k)}{\text{BesselY}(0, 0.1\cdot k)}, \frac{-\text{BesselJ}(0, 0.2\cdot k)}{\text{BesselY}(0, 0.2\cdot k)}\right\}, k=25..33, color=[red, blue]\right);$$



$$> fsolve\left(\frac{-\text{BesselJ}(0, 0.1\cdot k)}{\text{BesselY}(0, 0.1\cdot k)} = \frac{-\text{BesselJ}(0, 0.2\cdot k)}{\text{BesselY}(0, 0.2\cdot k)}, k, 30..33\right);$$

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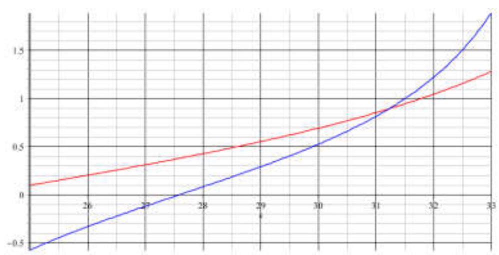

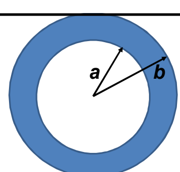

$$f(r) = A\left(J_m(k_{mn}r) - \frac{J_m(k_{mn}a)}{N_m(k_{mn}a)} N_m(k_{mn}r)\right) \qquad k_{01} = 31.230309$$