PHY 711 Classical Mechanics and Mathematical Methods
9-9:50 AM MWF Olin 107

Plan for Lecture 27:

Chap. 8 in F & W: Summary of two-dimensional membrane analysis

Chap. 9 in F & W: Introduction to hydrodynamics

1. **Motivation for topic**
2. **Newton's laws for fluids**
3. **Conservation relations**

19	Mon, 10/16/2017		Discuss exam questions and topics for presentations	Topic	10/18/2017
20	Wed, 10/18/2017	Chap. 7	Wave equation in one dimension	#11	10/20/2017
21	Fri, 10/20/2017	Chap. 7	Solutions of Sturm-Liouville equations	#12	10/27/2017
22	Mon, 10/23/2017	Chap. 7	Solutions of Sturm-Liouville equations		
23	Wed, 10/25/2017	Chap. 7	Solutions of Sturm-Liouville equations		
24	Fri, 10/27/2017	App. A	Laplace transforms and contour integrals	#13	11/01/2017
25	Mon, 10/30/2017	App. A	Contour integrals		
26	Wed, 11/01/2017	Chap. 8	Mechanics of Elastic Membranes	#14	11/06/2017
27	Fri, 11/03/2017	Chap. 9	Introduction to hydrodynamics		
28	Mon, 11/06/2017				
29	Wed, 11/08/2017				
30	Fri, 11/10/2017				
31	Mon, 11/13/2017				
32	Wed, 11/15/2017				
33	Fri, 11/17/2017				
34	Mon, 11/20/2017				
	Wed, 11/22/2017		*Thanksgiving Holiday -- No class*		
	Fri, 11/24/2017		*Thanksgiving Holiday -- No class*		
35	Mon, 11/27/2017				

Review --

Two-dimensional wave equation:

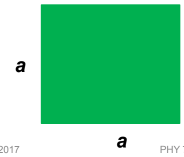
$$\frac{\partial^2 u}{\partial t^2} - c^2 \nabla^2 u = 0 \qquad \text{where } c^2 = \frac{\tau}{\sigma}$$

Standing wave solutions:

$$u(x,y,t) = \Re\left(e^{-i\omega t}\rho(x,y)\right)$$

$$\left(\nabla^2 + k^2\right)\rho(x,y) = 0 \qquad \text{where } k = \frac{\omega}{c}$$

Consider a square boundary:



Free boundary conditions:

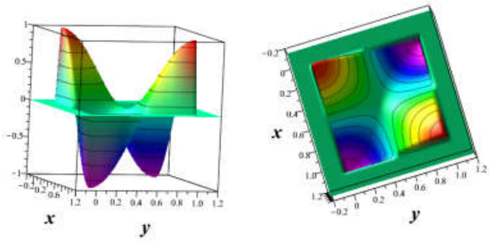
$$\frac{\partial \rho(0,y)}{\partial x} = \frac{\partial \rho(a,y)}{\partial x} = \frac{\partial \rho(x,a)}{\partial y} = \frac{\partial \rho(x,0)}{\partial y} = 0$$

$$\Rightarrow \rho_{mn}(x,y) = A\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$

$$k_{mn}^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \qquad \omega_{mn} = ck_{mn}$$

For $n = m = 1$:

$$\rho_{11}(x,y) = A\cos\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi y}{b}\right)$$



Hydrodynamic analysis

Motivation

1. Natural progression from strings, membranes, fluids; description of 1, 2, and 3 dimensional continua
2. Interesting and technologically important phenomena associated with fluids

Plan

1. Newton's laws for fluids
2. Continuity equation
3. Stress tensor
4. Energy relations
5. Bernoulli's theorem
6. Various examples
7. Sound waves

Newton's equations for fluids

Use Euler formulation; following "particles" of fluid

Variables: Density $\rho(x,y,z,t)$

Pressure $p(x,y,z,t)$

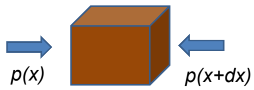
Velocity $\mathbf{v}(x,y,z,t)$

$$m\mathbf{a} = \mathbf{F}$$

$$m \rightarrow \rho dV$$

$$\mathbf{a} \rightarrow \frac{d\mathbf{v}}{dt}$$

$$\mathbf{F} \rightarrow \mathbf{F}_{applied} + \mathbf{F}_{pressure}$$



$$F_{pressure}\Big|_x = \left(-p(x+dx,y,z)+p(x,y,z)\right)dydz$$
$$= \frac{\left(-p(x+dx,y,z)+p(x,y,z)\right)}{dx}dxdydz$$
$$= -\frac{\partial p}{\partial x}dV$$

Newton's equations for fluids -- continued

$$m\mathbf{a} = \mathbf{F}_{applied} + \mathbf{F}_{pressure}$$
$$\rho dV\frac{d\mathbf{v}}{dt} = \mathbf{f}_{applied}\,\rho dV - (\nabla p)dV$$
$$\rho\frac{d\mathbf{v}}{dt} = \rho\mathbf{f}_{applied} - \nabla p$$

Detailed analysis of accelerati on term :

$$\mathbf{v} = \mathbf{v}(x,y,z,t)$$
$$\frac{d\mathbf{v}}{dt} = \frac{\partial\mathbf{v}}{\partial x}\frac{dx}{dt} + \frac{\partial\mathbf{v}}{\partial y}\frac{dy}{dt} + \frac{\partial\mathbf{v}}{\partial z}\frac{dz}{dt} + \frac{\partial\mathbf{v}}{\partial t}$$
$$\frac{d\mathbf{v}}{dt} = \frac{\partial\mathbf{v}}{\partial x}v_x + \frac{\partial\mathbf{v}}{\partial y}v_y + \frac{\partial\mathbf{v}}{\partial z}v_z + \frac{\partial\mathbf{v}}{\partial t}$$
$$\frac{d\mathbf{v}}{dt} = (\mathbf{v}\cdot\nabla)\mathbf{v} + \frac{\partial\mathbf{v}}{\partial t}$$

Note that :

$$\frac{\partial\mathbf{v}}{\partial x}v_x + \frac{\partial\mathbf{v}}{\partial y}v_y + \frac{\partial\mathbf{v}}{\partial z}v_z = \nabla\left(\frac{1}{2}\mathbf{v}\cdot\mathbf{v}\right) - \mathbf{v}\times(\nabla\times\mathbf{v})$$

Newton's equations for fluids -- continued

$$\rho \frac{d\mathbf{v}}{dt} = \rho \mathbf{f}_{applied} - \nabla p$$

$$\rho\left(\nabla\left(\frac{1}{2}\mathbf{v}\cdot\mathbf{v}\right) - \mathbf{v}\times(\nabla\times\mathbf{v}) + \frac{\partial \mathbf{v}}{\partial t}\right) = \rho \mathbf{f}_{applied} - \nabla p$$

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) - \mathbf{v}\times(\nabla\times\mathbf{v}) = \mathbf{f}_{applied} - \frac{\nabla p}{\rho}$$

Solution of Euler's equation for fluids

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) - \mathbf{v}\times(\nabla\times\mathbf{v}) = \mathbf{f}_{applied} - \frac{\nabla p}{\rho}$$

Consider the following restrictions:

1. $(\nabla\times\mathbf{v}) = 0$ "irrotational flow"

   $\Rightarrow \mathbf{v} = -\nabla\Phi$

2. $\mathbf{f}_{applied} = -\nabla U$ conservative applied force
3. $\rho = \text{(constant)}$ incompressible fluid

$$\frac{\partial(-\nabla\Phi)}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) = -\nabla U - \frac{\nabla p}{\rho}$$

$$\Rightarrow \nabla\left(\frac{p}{\rho} + U + \tfrac{1}{2}v^2 - \frac{\partial\Phi}{\partial t}\right) = 0$$

Bernoulli's integral of Euler's equation

$$\nabla\left(\frac{p}{\rho} + U + \tfrac{1}{2}v^2 - \frac{\partial\Phi}{\partial t}\right) = 0$$

Integrating over space:

$$\frac{p}{\rho} + U + \tfrac{1}{2}v^2 - \frac{\partial\Phi}{\partial t} = C(t)$$

where $\mathbf{v} = -\nabla\Phi(\mathbf{r},t) = -\nabla\left(\Phi(\mathbf{r},t) + C'(t)\right)$

$$\Rightarrow \frac{p}{\rho} + U + \tfrac{1}{2}v^2 - \frac{\partial\Phi}{\partial t} = 0 \qquad \text{Bernoulli's theorem}$$

Examples of Bernoulli's theorem

$$\frac{p}{\rho}+U+\tfrac{1}{2}v^2-\frac{\partial\Phi}{\partial t}=0$$

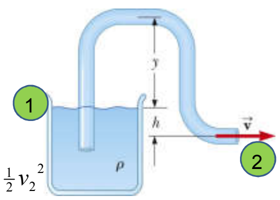
Modified form; assuming $\frac{\partial\Phi}{\partial t}=0$

$$\frac{p}{\rho}+U+\tfrac{1}{2}v^2=\text{constant}$$

$$p_1=p_2=p_{atm}$$

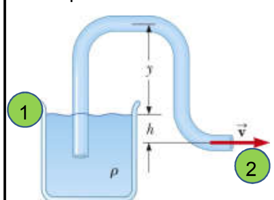
$$U_1-U_2=gh$$

$$v_1\approx 0$$



$$\frac{p_1}{\rho}+U_1+\tfrac{1}{2}v_1^2=\frac{p_2}{\rho}+U_2+\tfrac{1}{2}v_2^2$$

Examples of Bernoulli's theorem -- continued



$$p_1=p_2=p_{atm}$$

$$U_1-U_2=gh$$

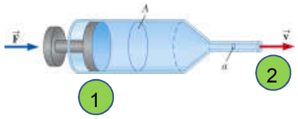
$$v_1\approx 0$$

$$\frac{p_1}{\rho}+U_1+\tfrac{1}{2}v_1^2=\frac{p_2}{\rho}+U_2+\tfrac{1}{2}v_2^2$$

$$v_2\approx\sqrt{2gh}$$

Examples of Bernoulli's theorem -- continued

$$\frac{p}{\rho}+U+\tfrac{1}{2}v^2=\text{constant}$$



$$p_1=\frac{F}{A}+p_{atm} \qquad p_2=p_{atm}$$

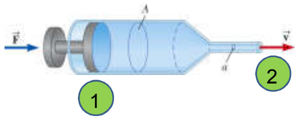
$$U_1=U_2$$

$$v_1A=v_2a \qquad \text{continuity equation}$$

$$\frac{p_1}{\rho}+U_1+\tfrac{1}{2}v_1^2=\frac{p_2}{\rho}+U_2+\tfrac{1}{2}v_2^2$$

Examples of Bernoulli's theorem -- continued

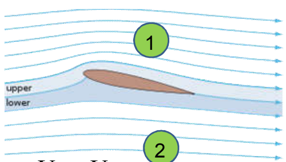
$$\frac{p}{\rho} + U + \tfrac{1}{2}v^2 = \text{constant}$$



$$\frac{2F}{A} = v_2^{\,2}\left(1 - \left(\frac{a}{A}\right)^2\right)$$

$$v_2 = \sqrt{\frac{2F/A}{1 - \left(\frac{a}{A}\right)^2}}$$

Examples of Bernoulli's theorem – continued
Approximate explanation of airplane lift
Cross section view of airplane wing
http://en.wikipedia.org/wiki/Lift_%28force%29



$$U_1 \approx U_2$$

$$\frac{p_1}{\rho} + U_1 + \tfrac{1}{2}v_1^{\,2} = \frac{p_2}{\rho} + U_2 + \tfrac{1}{2}v_2^{\,2}$$

$$p_2 - p_1 = \tfrac{1}{2}\rho\left(v_1^{\,2} - v_2^{\,2}\right)$$

Continuity equation connecting fluid density and velocity:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial \rho}{\partial t} + \rho(\nabla \cdot \mathbf{v}) + (\nabla \rho) \cdot \mathbf{v} = 0$$

$$\text{Consider: } \frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + (\nabla \rho) \cdot \mathbf{v}$$

$$\Rightarrow \frac{d\rho}{dt} + \rho(\nabla \cdot \mathbf{v}) = 0 \qquad \text{alternative form of continuity equation}$$

Some details on the velocity potential

Continuity equation :

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial \rho}{\partial t} + \rho(\nabla \cdot \mathbf{v}) + (\nabla \rho) \cdot \mathbf{v} = 0$$

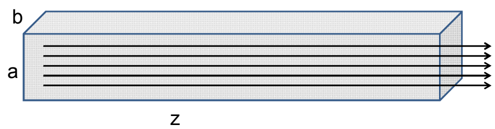
For incompressible fluid : $\rho = (\text{constant})$

$\Rightarrow \nabla \cdot \mathbf{v} = 0$

Irrotational flow : $\nabla \times \mathbf{v} = 0$ $\Rightarrow \mathbf{v} = -\nabla \Phi$

$\Rightarrow \nabla^2 \Phi = 0$

Example – uniform flow



$$\nabla^2 \Phi = 0$$

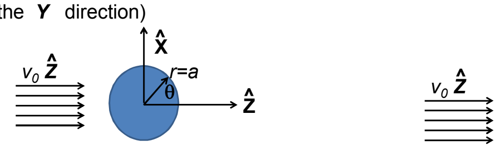
$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

Possible solution :

$\Phi = -v_o z$

$\mathbf{v} = -\nabla \Phi = v_o \hat{\mathbf{z}}$

Example – flow around a long cylinder (oriented in the ***Y*** direction)



$$\nabla^2 \Phi = 0$$

$$\left.\frac{\partial \Phi}{\partial r}\right|_{r=a} = 0$$

Laplace equation in cylindrical coordinates

($r, \theta,$ defined in x-z plane; y representing cylinder axis)

$$\nabla^2\Phi = 0 = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} + \frac{\partial^2 \Phi}{\partial y^2}$$

In our case, there is no motion in the y dimension

$$\Rightarrow \Phi(r,\theta,y) = \Phi(r,\theta)$$

From boundary condition : $v_z(r \to \infty) = v_0$

$$\frac{\partial \Phi}{\partial z}(r \to \infty) = -v_0 \qquad \Rightarrow \Phi(r \to \infty, \theta) = -v_0 r \cos\theta$$

Note that : $\frac{\partial^2 \cos\theta}{\partial \theta^2} = -\cos\theta$

Guess form : $\Phi(r,\theta) = f(r)\cos\theta$

Necessary equation for radial function

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial f}{\partial r} - \frac{1}{r^2} f = 0$$

$$f(r) = Ar + \frac{B}{r} \qquad \text{where } A, B \text{ are constants}$$

Boundary condition on cylinder surface :

$$\left.\frac{\partial \Phi}{\partial r}\right|_{r=a} = 0$$

$$\frac{df}{dr}(r=a) = 0 = A - \frac{B}{a^2}$$

$$\Rightarrow B = Aa^2$$

Boundary condition at ∞ : $\Rightarrow A = -v_0$

$$\Phi(r,\theta) = -v_0\left(r + \frac{a^2}{r}\right)\cos\theta$$

$$v_r = -\frac{\partial \Phi}{\partial r} = v_0\left(1 - \frac{a^2}{r^2}\right)\cos\theta$$

$$v_\theta = -\frac{1}{r}\frac{\partial \Phi}{\partial \theta} = v_0\left(1 + \frac{a^2}{r^2}\right)\sin\theta$$

For 3-dimensional system, consider a spherical obstruction

Laplacian in spherical polar coordinates:

$$\nabla^2\Phi = 0 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \Phi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \Phi}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \Phi}{\partial \varphi^2}$$

to be continued …