PHY 711 Classical Mechanics and Mathematical Methods
9-9:50 AM MWF Olin 107

Plan for Lecture 28:

Introduction to hydrodynamics

(Chap. 9 in F & W)

1. **Incompressible fluids**
2. **Isentropic fluids**

	Fri, 10/13/2017		*Fall break -- No class*		
19	Mon, 10/16/2017		Discuss exam questions and topics for presentations	Topic	10/18/2017
20	Wed, 10/18/2017	Chap. 7	Wave equation in one dimension	#11	10/20/2017
21	Fri, 10/20/2017	Chap. 7	Solutions of Sturm-Liouville equations	#12	10/27/2017
22	Mon, 10/23/2017	Chap. 7	Solutions of Sturm-Liouville equations		
23	Wed, 10/25/2017	Chap. 7	Solutions of Sturm-Liouville equations		
24	Fri, 10/27/2017	App. A	Laplace transforms and contour integrals	#13	11/01/2017
25	Mon, 10/30/2017	App. A	Contour integrals		
26	Wed, 11/01/2017	Chap. 8	Mechanics of Elastic Membranes	#14	11/06/2017
27	Fri, 11/03/2017	Chap. 9	Introduction to hydrodynamics		
28	Mon, 11/06/2017	Chap. 9	Introduction to hydrodynamics	#15	11/10/2017
29	Wed, 11/08/2017				
30	Fri, 11/10/2017				
31	Mon, 11/13/2017				
32	Wed, 11/15/2017				
33	Fri, 11/17/2017				
34	Mon, 11/20/2017				
	Wed, 11/22/2017		*Thanksgiving Holiday -- No class*		
	Fri, 11/24/2017		*Thanksgiving Holiday -- No class*		
35	Mon, 11/27/2017				

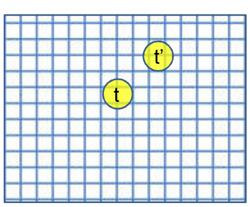
Newton's equations for fluids

Use **Euler** formulation; properties described in terms of stationary spatial grid

$$\text{Variables:}\quad \text{Density}\quad \rho(x,y,z,t)$$

$$\text{Pressure}\quad p(x,y,z,t)$$

$$\text{Velocity}\quad \mathbf{v}(x,y,z,t)$$



$$\text{Particle at } t:\quad \mathbf{r}, t$$

$$\text{Particle at } t':\quad \mathbf{r}+\mathbf{v}\delta t, t'$$

$$t' = t + \delta t$$

Euler analysis -- continued

Particle at t: $\mathbf{r},t$

Particle at t': $\mathbf{r}+\mathbf{v}\delta t, t'$ where $\delta t = t' - t$

For $f(\mathbf{r},t)$:

$$\frac{df}{dt} = \lim_{\delta t \to 0}\left(\frac{f(\mathbf{r},t') - f(\mathbf{r},t)}{\delta t} + \frac{f(\mathbf{r}+\mathbf{v}\delta t,t) - f(\mathbf{r},t)}{\delta t}\right)$$

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + (\mathbf{v}\cdot\nabla) f$$

Example: $(\mathbf{v}\cdot\nabla)\mathbf{v} = \nabla\left(\frac{1}{2}v^2\right) - \mathbf{v}\times(\nabla\times\mathbf{v})$

Some details on the velocity potential

Continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0$$

$$\frac{\partial \rho}{\partial t} + \rho(\nabla\cdot\mathbf{v}) + (\nabla\rho)\cdot\mathbf{v} = 0$$

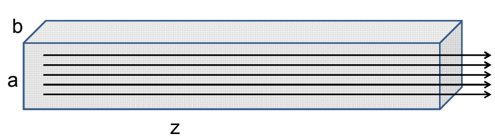
For incompressible fluid: $\rho = (\text{constant})$

$\Rightarrow \nabla\cdot\mathbf{v} = 0$

Irrotational flow: $\nabla\times\mathbf{v} = 0$ $\Rightarrow \mathbf{v} = -\nabla\Phi$

$\Rightarrow \nabla^2\Phi = 0$

Example – uniform flow



$$\nabla^2\Phi = 0$$

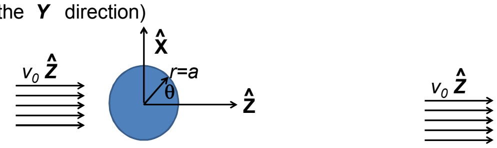
$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = 0$$

Possible solution:

$\Phi = -v_o z$

$\mathbf{v} = -\nabla\Phi = v_o\hat{\mathbf{z}}$

Example – flow around a long cylinder (oriented in the ***Y*** direction)



$$\nabla^2\Phi = 0$$

$$\left.\frac{\partial\Phi}{\partial r}\right|_{r=a} = 0$$

Laplace equation in cylindrical coordinates

($r,\theta,$ defined in x-z plane; y representing cylinder axis)

$$\nabla^2\Phi = 0 = \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} + \frac{\partial^2\Phi}{\partial y^2}$$

In our case, there is no motion in the y dimension

$$\Rightarrow \Phi(r,\theta,y) = \Phi(r,\theta)$$

From boundary condition : $v_z(r\rightarrow\infty) = v_0$

$$\frac{\partial\Phi}{\partial z}(r\rightarrow\infty) = -v_0 \qquad \Rightarrow \Phi(r\rightarrow\infty,\theta) = -v_0 r\cos\theta$$

Note that : $\dfrac{\partial^2\cos\theta}{\partial\theta^2} = -\cos\theta$

Guess form : $\Phi(r,\theta) = f(r)\cos\theta$

Necessary equation for radial function

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial f}{\partial r} - \frac{1}{r^2}f = 0$$

$$f(r) = Ar + \frac{B}{r} \qquad \text{where } A, B \text{ are constants}$$

Boundary condition on cylinder surface :

$$\left.\frac{\partial\Phi}{\partial r}\right|_{r=a} = 0$$

$$\frac{df}{dr}(r=a) = 0 = A - \frac{B}{a^2}$$

$$\Rightarrow B = Aa^2$$

Boundary condition at ∞ : $\Rightarrow A = -v_0$

$$\Phi(r,\theta) = -v_0\left(r + \frac{a^2}{r}\right)\cos\theta$$

$$v_r = -\frac{\partial\Phi}{\partial r} = v_0\left(1 - \frac{a^2}{r^2}\right)\cos\theta$$

$$v_\theta = -\frac{1}{r}\frac{\partial\Phi}{\partial\theta} = -v_0\left(1 + \frac{a^2}{r^2}\right)\sin\theta$$

For 3-dimensional system, consider a spherical obstruction

Laplacian in spherical polar coordinates:

$$\nabla^2\Phi = 0 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\varphi^2}$$

Spherical system continued:

Laplacian in spherical polar coordinates:

$$\nabla^2\Phi = 0 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\varphi^2}$$

In terms of spherical harmonic functions:

$$\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)Y_{lm}(\theta,\phi) = -l(l+1)Y_{lm}(\theta,\phi)$$

In our case:

$$Y_{10}(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$\Phi(r,\theta,\phi) = f(r)Y_{lm}(\theta,\phi)$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{df}{dr}\right) - \frac{l(l+1)}{r^2}f = 0$$

(Continue analysis for homework)

Solution of Euler's equation for fluids

$$\frac{\partial\mathbf{v}}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) - \mathbf{v}\times(\nabla\times\mathbf{v}) = \mathbf{f}_{applied} - \frac{\nabla p}{\rho}$$

Consider the following restrictions:

1. $(\nabla\times\mathbf{v}) = 0$ "irrotational flow"

   $\Rightarrow \mathbf{v} = -\nabla\Phi$

2. $\mathbf{f}_{applied} = -\nabla U$ conservative applied force

3. $\rho =$ (constant) incompressible fluid

$$\frac{\partial(-\nabla\Phi)}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) = -\nabla U - \frac{\nabla p}{\rho}$$

$$\Rightarrow \nabla\left(\frac{p}{\rho} + U + \tfrac{1}{2}v^2 - \frac{\partial\Phi}{\partial t}\right) = 0$$

Bernoulli's integral of Euler's equation for constant ρ

$$\nabla\left(\frac{p}{\rho}+U+\tfrac{1}{2}v^2-\frac{\partial\Phi}{\partial t}\right)=0$$

Integrating over space:

$$\frac{p}{\rho}+U+\tfrac{1}{2}v^2-\frac{\partial\Phi}{\partial t}=C(t)$$

where $\mathbf{v}=-\nabla\Phi(\mathbf{r},t)=-\nabla\big(\Phi(\mathbf{r},t)+C(t)\big)$

$$\Rightarrow \frac{p}{\rho}+U+\tfrac{1}{2}v^2-\frac{\partial\Phi}{\partial t}=C_0 \qquad \text{Bernoulli's theorem}$$

For incompressible fluid

Solution of Euler's equation for fluids -- isentropic

$$\frac{\partial\mathbf{v}}{\partial t}+\nabla\left(\tfrac{1}{2}v^2\right)-\mathbf{v}\times\left(\nabla\times\mathbf{v}\right)=\mathbf{f}_{applied}-\frac{\nabla p}{\rho}$$

Consider the following restrictions:

1. $(\nabla\times\mathbf{v})=0$ "irrotational flow"

   $\Rightarrow \mathbf{v}=-\nabla\Phi$

2. $\mathbf{f}_{applied}=-\nabla U$ conservative applied force
3. $\rho\neq$ (constant) isentropic fluid

A little thermodynamics

First law of thermodynamics: $dE_{\text{int}}=dQ-dW$

For isentropic conditions: $dQ=0$

$dE_{\text{int}}=-dW=-pdV$

Solution of Euler's equation for fluids – isentropic (continued)

$dE_{\text{int}}=-dW=-pdV$

In terms of mass density: $\rho=\dfrac{M}{V}$

For fixed M and variable V: $d\rho=-\dfrac{M}{V^2}dV$

$$dV=-\frac{M}{\rho^2}d\rho$$

In terms in intensive variables: Let $E_{\text{int}}=M\varepsilon$

$$dE_{\text{int}}=Md\varepsilon=-dW=-pdV=M\frac{p}{\rho^2}d\rho$$

$$d\varepsilon=\frac{p}{\rho^2}d\rho \qquad \left(\frac{\partial\varepsilon}{\partial\rho}\right)_{dQ=0}=\frac{p}{\rho^2}$$

Solution of Euler's equation for fluids – isentropic (continued)

$$\left(\frac{\partial \varepsilon}{\partial \rho}\right)_{dQ=0} = \frac{p}{\rho^2}$$

$$\text{Consider}: \quad \nabla\varepsilon = \left(\frac{\partial \varepsilon}{\partial \rho}\right)_{dQ=0} \nabla\rho = \frac{p}{\rho^2}\nabla\rho$$

$$\text{Rearranging}: \quad \nabla\left(\varepsilon + \frac{p}{\rho}\right) = \frac{\nabla p}{\rho}$$

Solution of Euler's equation for fluids – isentropic (continued)

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) - \mathbf{v}\times(\nabla\times\mathbf{v}) = \mathbf{f}_{applied} - \frac{\nabla p}{\rho}$$

$$\frac{\nabla p}{\rho} = \nabla\left(\varepsilon + \frac{p}{\rho}\right)$$

$$\nabla\times\mathbf{v} = 0 \qquad \mathbf{v} = -\nabla\Phi \qquad \mathbf{f}_{applied} = -\nabla U$$

$$\frac{\partial(-\nabla\Phi)}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) = -\nabla U - \nabla\left(\varepsilon + \frac{p}{\rho}\right)$$

$$\Rightarrow \nabla\left(\varepsilon + \frac{p}{\rho} + U + \tfrac{1}{2}v^2 - \frac{\partial\Phi}{\partial t}\right) = 0$$

Summary of Bernoulli's results

For incompressible fluid

$$\nabla\left(\frac{p}{\rho} + U + \tfrac{1}{2}v^2 - \frac{\partial\Phi}{\partial t}\right) = 0$$

For isentropic fluid with internal energy density ε

$$\nabla\left(\varepsilon + \frac{p}{\rho} + U + \tfrac{1}{2}v^2 - \frac{\partial\Phi}{\partial t}\right) = 0$$

Application of fluid equations to the case of air in equilibrium plus small perturbation

Newton - Euler equation of motion :

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{f}_{applied} - \frac{\nabla p}{\rho}$$

Continuity equation : $\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0$

Near equilibrium :

$$\rho = \rho_0 + \delta\rho$$
$$p = p_0 + \delta p$$
$$\mathbf{v} = 0 + \delta\mathbf{v}$$
$$\mathbf{f}_{applied} = 0$$

Equations to lowest order in perturbation :

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{f}_{applied} - \frac{\nabla p}{\rho} \quad \Rightarrow \quad \frac{\partial \delta\mathbf{v}}{\partial t} = -\frac{\nabla \delta p}{\rho_0}$$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0 \quad \Rightarrow \quad \frac{\partial \delta\rho}{\partial t} + \rho_0 \nabla\cdot\delta\mathbf{v} = 0$$

In terms of the velocity potential :

$$\delta\mathbf{v} = -\nabla\Phi$$

$$\frac{\partial \delta\mathbf{v}}{\partial t} = -\frac{\nabla \delta p}{\rho_0} \quad \Rightarrow \nabla\left(-\frac{\partial\Phi}{\partial t} + \frac{\delta p}{\rho_0}\right) = 0$$

$$\frac{\partial \delta\rho}{\partial t} + \rho_0 \nabla\cdot\delta\mathbf{v} = 0 \quad \Rightarrow \frac{\partial \delta\rho}{\partial t} - \rho_0\nabla^2\Phi = 0$$

Expressing pressure in terms of the density :

$p = p(s,\rho) = p_0 + \delta p$ where s denotes the (constant) entropy

$$p_0 = p(s,\rho_0)$$

$$\delta p = \left(\frac{\partial p}{\partial \rho}\right)_s \delta\rho \equiv c^2 \delta\rho$$

$$\nabla\left(-\frac{\partial\Phi}{\partial t} + \frac{\delta p}{\rho_0}\right) = 0 \quad \Rightarrow -\frac{\partial\Phi}{\partial t} + c^2\frac{\delta\rho}{\rho_0} = (\text{constant})$$

$$\Rightarrow -\frac{\partial^2\Phi}{\partial t^2} + \frac{c^2}{\rho_0}\frac{\partial\delta\rho}{\partial t} = 0$$

$$\frac{\partial \delta\rho}{\partial t} - \rho_0\nabla^2\Phi = 0 \quad \Rightarrow \frac{\partial^2\Phi}{\partial t^2} - c^2\nabla^2\Phi = 0$$

Wave equation for air :

$$\frac{\partial^2 \Phi}{\partial t^2} - c^2 \nabla^2 \Phi = 0$$

Here, $c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s$

$\mathbf{v} = -\nabla \Phi$

Note that, we also have :

$$\frac{\partial^2 \delta\rho}{\partial t^2} - c^2 \nabla^2 \delta\rho = 0$$

$$\frac{\partial^2 \delta p}{\partial t^2} - c^2 \nabla^2 \delta p = 0$$

Boundary values:

Impenetrable surface with normal $\hat{\mathbf{n}}$ moving at velocity $\mathbf{V}$:

$$\hat{\mathbf{n}} \cdot \mathbf{V} = \hat{\mathbf{n}} \cdot \delta\mathbf{v} = -\hat{\mathbf{n}} \cdot \nabla \Phi$$

Free surface:

$$\delta p = 0 \qquad \Rightarrow \rho_0 \frac{\partial \Phi}{\partial t} = 0$$

Analysis of wave velocity in an ideal gas:

$$c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s$$

Equation of state for ideal gas :

$$pV = NkT \qquad N = \frac{M}{M_0}$$

$$p = \frac{M}{V}\frac{k}{M_0}T = \rho \frac{k}{M_0}T$$

$k = 1.38 \times 10^{-23} J/k$

M_0 = average mass of each molecule

Internal energy for ideal gas :

$$E = \frac{f}{2}NkT = M\varepsilon \qquad \varepsilon = \frac{f}{2}\frac{k}{M_0}T = \frac{f}{2}\frac{p}{\rho}$$

In terms of specific heat ratio : $\gamma \equiv \frac{C_p}{C_V}$

$$dE = dQ - dW$$

$$C_V = \left(\frac{dQ}{dT}\right)_V = \left(\frac{\partial E}{\partial T}\right)_V = \frac{f}{2}\frac{Mk}{M_0}$$

$$C_p = \left(\frac{dQ}{dT}\right)_p = \left(\frac{\partial E}{\partial T}\right)_p + p\left(\frac{\partial V}{\partial T}\right)_p = \frac{f}{2}\frac{Mk}{M_0} + \frac{Mk}{M_0}$$

$$\gamma = \frac{C_p}{C_V} = \frac{\frac{f}{2}+1}{\frac{f}{2}} \qquad \Rightarrow \frac{f}{2} = \frac{1}{\gamma - 1}$$

Internal energy for ideal gas :

$$E = \frac{1}{\gamma - 1} NkT = M\varepsilon \qquad \varepsilon = \frac{1}{\gamma - 1}\frac{k}{M_0}T = \frac{1}{\gamma - 1}\frac{p}{\rho}$$

Internal energy for ideal gas under isentropic conditions :

$$d\varepsilon = -\frac{p}{M}dV = \frac{p}{\rho^2}d\rho$$

$$\left(\frac{\partial \varepsilon}{\partial \rho}\right)_s = \frac{p}{\rho^2} = \frac{\partial}{\partial \rho}\left(\frac{1}{\gamma - 1}\frac{p}{\rho}\right)_s = \left(\frac{\partial p}{\partial \rho}\right)_s \frac{1}{(\gamma - 1)\rho} - \frac{p}{(\gamma - 1)\rho^2}$$

$$\Rightarrow \left(\frac{\partial p}{\partial \rho}\right)_s = \frac{p\gamma}{\rho}$$

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Alternative derivation :

Isentropic or adiabatic equation of state :

$$\frac{dp}{p} = \gamma \frac{d\rho}{\rho} \qquad \Rightarrow \frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^\gamma$$

$$\left(\frac{\partial p}{\partial \rho}\right)_s = \frac{p\gamma}{\rho}$$

Linearized speed of sound

$$c_0{}^2 = \left(\frac{\partial p}{\partial \rho}\right)_{s, p_0, \rho_0} = \frac{p_0\gamma}{\rho_0}$$

$$c_0{}^2 \approx \frac{1.5 \cdot 1.013 \times 10^5\, Pa}{1.3 kg/m^3} \qquad c_0 \approx 340 \text{ m/s}$$

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Density dependence of speed of sound for ideal gas :

$$c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s = \frac{p\gamma}{\rho}$$

$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^\gamma$$

$$c^2 = \frac{p_0\gamma}{\rho_0}\frac{p/p_0}{\rho/\rho_0} = c_0^2\left(\frac{\rho}{\rho_0}\right)^{\gamma - 1}$$

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