

**PHY 711 Classical Mechanics and
Mathematical Methods
9-9:50 AM MWF Olin 107**

Plan for Lecture 30: Chap. 9 of F&W

**Wave equation for sound in the linear
approximation**

- 1. Sound generation**
- 2. Sound scattering**

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23	Wed, 10/25/2017	Chap. 7	Solutions of Sturm-Liouville equations		
24	Fri, 10/27/2017	App. A	Laplace transforms and contour integrals	#13	11/01/2017
25	Mon, 10/30/2017	App. A	Contour integrals		
26	Wed, 11/01/2017	Chap. 8	Mechanics of Elastic Membranes	#14	11/06/2017
27	Fri, 11/03/2017	Chap. 9	Introduction to hydrodynamics		
28	Mon, 11/06/2017	Chap. 9	Introduction to hydrodynamics	#15	11/10/2017
29	Wed, 11/08/2017	Chap. 9	Sound waves		
30	Fri, 11/10/2017	Chap. 9	Sound waves	#16	11/17/2017
31	Mon, 11/13/2017				
32	Wed, 11/15/2017				
33	Fri, 11/17/2017				
34	Mon, 11/20/2017				
	Wed, 11/22/2017		Thanksgiving Holiday -- No class		
	Fri, 11/24/2017		Thanksgiving Holiday -- No class		
35	Mon, 11/27/2017				
36	Wed, 11/29/2017				
37	Fri, 12/01/2017				
38	Mon, 12/04/2017		Review	Prepare presentations	

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Solutions to wave equation:

$$\nabla^2 \Phi - \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} = 0$$

Plane wave solution :

$$\Phi(\mathbf{r}, t) = A e^{i\mathbf{k} \cdot \mathbf{r} - i\omega t} \quad \text{where} \quad k^2 = \left(\frac{\omega}{c} \right)^2$$

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Wave equation with source:

$$\nabla^2 \Phi - \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} = -f(\mathbf{r}, t)$$

Solution in terms of Green's function :

$$\Phi(\mathbf{r}, t) = \int d^3 r' \int dt' G(\mathbf{r} - \mathbf{r}', t - t') f(\mathbf{r}', t')$$

where

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) G(\mathbf{r} - \mathbf{r}', t - t') = -\delta(\mathbf{r} - \mathbf{r}') \delta(t - t')$$

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Wave equation with source -- continued:

We can show that :

$$G(\mathbf{r} - \mathbf{r}', t - t') = \frac{\delta\left(t' - \left(t \mp \frac{|\mathbf{r} - \mathbf{r}'|}{c}\right)\right)}{4\pi|\mathbf{r} - \mathbf{r}'|}$$

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Derivation of Green's function for wave equation

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) G(\mathbf{r} - \mathbf{r}', t - t') = -\delta(\mathbf{r} - \mathbf{r}') \delta(t - t')$$

Recall that

$$\delta(t - t') = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega(t-t')} d\omega$$

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Derivation of Green's function for wave equation -- continued

Define: $\tilde{G}(\mathbf{r}, \omega) = \int_{-\infty}^{\infty} G(\mathbf{r}, t) e^{i\omega t} dt$

$$G(\mathbf{r}, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{G}(\mathbf{r}, \omega) e^{-i\omega t} d\omega$$

$\tilde{G}(\mathbf{r}, \omega)$ must satisfy:

$$(\nabla^2 + k^2) \tilde{G}(\mathbf{r} - \mathbf{r}', \omega) = -\delta(\mathbf{r} - \mathbf{r}') \quad \text{where } k^2 = \frac{\omega^2}{c^2}$$

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Derivation of Green's function for wave equation -- continued

$$(\nabla^2 + k^2) \tilde{G}(\mathbf{r} - \mathbf{r}', \omega) = -\delta(\mathbf{r} - \mathbf{r}')$$

Solution assuming isotropy in $\mathbf{r} - \mathbf{r}'$:

$$\tilde{G}(\mathbf{r} - \mathbf{r}', \omega) = \frac{e^{\pm ik|\mathbf{r} - \mathbf{r}'|}}{4\pi|\mathbf{r} - \mathbf{r}'|}$$

Check -- Define $R \equiv |\mathbf{r} - \mathbf{r}'|$ and for $R > 0$:

$$(\nabla^2 + k^2) \tilde{G}(R, \omega) = \frac{1}{R} \frac{d^2}{dR^2} (R \tilde{G}(R, \omega)) + k^2 \tilde{G}(R, \omega) = 0$$

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Derivation of Green's function for wave equation -- continued

For $R > 0$:

$$(\nabla^2 + k^2) \tilde{G}(R, \omega) = \frac{1}{R} \frac{d^2}{dR^2} (R \tilde{G}(R, \omega)) + k^2 \tilde{G}(R, \omega) = 0$$

$$\frac{d^2}{dR^2} (R \tilde{G}(R, \omega)) + k^2 (R \tilde{G}(R, \omega)) = 0$$

$$(R \tilde{G}(R, \omega)) = A e^{ikR} + B e^{-ikR}$$

$$\Rightarrow \tilde{G}(R, \omega) = A \frac{e^{ikR}}{R} + B \frac{e^{-ikR}}{R}$$

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Derivation of Green's function for wave equation – continued
need to find A and B.

Note that : $\nabla^2 \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|} = -\delta(\mathbf{r}-\mathbf{r}')$

$$\Rightarrow A = B = \frac{1}{4\pi}$$

$$\tilde{G}(R, \omega) = \frac{e^{\pm ikR}}{4\pi R}$$

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Derivation of Green's function for wave equation – continued

$$\begin{aligned} G(\mathbf{r}-\mathbf{r}', t-t') &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{G}(\mathbf{r}-\mathbf{r}', \omega) e^{-i\omega(t-t')} d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{\pm ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} e^{-i\omega(t-t')} d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{\pm i\frac{\omega}{c}|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} e^{-i\omega(t-t')} d\omega \end{aligned}$$

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Derivation of Green's function for wave equation – continued

$$G(\mathbf{r}-\mathbf{r}', t-t') = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{\pm i\frac{\omega}{c}|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} e^{-i\omega(t-t')} d\omega$$

Noting that $\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega u} d\omega = \delta(u)$

$$\Rightarrow G(\mathbf{r}-\mathbf{r}', t-t') = \frac{\delta\left(t - \left(t' \mp \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)\right)}{4\pi|\mathbf{r}-\mathbf{r}'|}$$

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→ In order to solve an inhomogeneous wave equation with a time harmonic forcing term, we can use the corresponding Green's function:

$$\tilde{G}(\mathbf{r}-\mathbf{r}',\omega) = \frac{e^{iik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}$$

In fact, this Green's function is appropriate for solving equations with boundary conditions at infinity. For solving problems with surface boundary conditions where we know the boundary values or their gradients, the Green's function must be modified.

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Green's theorem

Consider two functions $h(\mathbf{r})$ and $g(\mathbf{r})$

Note that: $\int_V (h\nabla^2 g - g\nabla^2 h) d^3r = \oint_S (h\nabla g - g\nabla h) \cdot \hat{\mathbf{n}} d^2r$

$$\nabla^2 \tilde{\Phi} + k^2 \tilde{\Phi} = -f(\mathbf{r}, \omega)$$

$$(\nabla^2 + k^2) \tilde{G}(\mathbf{r}-\mathbf{r}', \omega) = -\delta(\mathbf{r}-\mathbf{r}')$$

$$h \leftrightarrow \tilde{\Phi}; \quad g \leftrightarrow \tilde{G}$$

$$\int_V (\tilde{\Phi}(\mathbf{r}, \omega) \delta(\mathbf{r}-\mathbf{r}') - \tilde{G}(\mathbf{r}-\mathbf{r}', \omega) f(\mathbf{r}, \omega)) d^3r =$$

$$\oint_S (\tilde{\Phi}(\mathbf{r}, \omega) \nabla \tilde{G}(\mathbf{r}-\mathbf{r}', \omega) - \tilde{G}(\mathbf{r}-\mathbf{r}', \omega) \nabla \tilde{\Phi}(\mathbf{r}, \omega)) \cdot \hat{\mathbf{n}} d^2r$$

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$$\int_V (\tilde{\Phi}(\mathbf{r}, \omega) \delta(\mathbf{r}-\mathbf{r}') - \tilde{G}(\mathbf{r}-\mathbf{r}', \omega) f(\mathbf{r}, \omega)) d^3r =$$

$$\oint_S (\tilde{\Phi}(\mathbf{r}, \omega) \nabla \tilde{G}(\mathbf{r}-\mathbf{r}', \omega) - \tilde{G}(\mathbf{r}-\mathbf{r}', \omega) \nabla \tilde{\Phi}(\mathbf{r}, \omega)) \cdot \hat{\mathbf{n}} d^2r$$

Exchanging $\mathbf{r} \leftrightarrow \mathbf{r}'$:

$$\int_V (\tilde{\Phi}(\mathbf{r}', \omega) \delta(\mathbf{r}-\mathbf{r}') - \tilde{G}(\mathbf{r}-\mathbf{r}', \omega) f(\mathbf{r}', \omega)) d^3r' =$$

$$\oint_S (\tilde{\Phi}(\mathbf{r}', \omega) \nabla \tilde{G}(\mathbf{r}-\mathbf{r}', \omega) - \tilde{G}(\mathbf{r}-\mathbf{r}', \omega) \nabla \tilde{\Phi}(\mathbf{r}', \omega)) \cdot \hat{\mathbf{n}} d^2r'$$

If the integration volume V includes the point $\mathbf{r} = \mathbf{r}'$:

$$\tilde{\Phi}(\mathbf{r}, \omega) = \int_V \tilde{G}(\mathbf{r}-\mathbf{r}', \omega) f(\mathbf{r}', \omega) d^3r' +$$

$$\oint_S (\tilde{\Phi}(\mathbf{r}', \omega) \nabla \tilde{G}(\mathbf{r}-\mathbf{r}', \omega) - \tilde{G}(\mathbf{r}-\mathbf{r}', \omega) \nabla \tilde{\Phi}(\mathbf{r}', \omega)) \cdot \hat{\mathbf{n}} d^2r'$$

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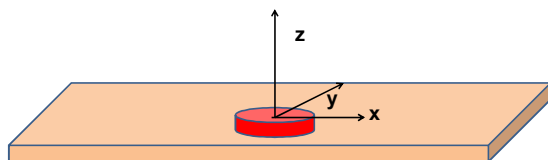
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Wave equation with source:

$$\nabla^2 \Phi - \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} = -f(\mathbf{r}, t)$$

Example:

$f(\mathbf{r}, t) \Rightarrow$ time harmonic piston of radius a , amplitude $\varepsilon \hat{\mathbf{z}}$
can be represented as boundary value of $\Phi(\mathbf{r}, t)$



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Treatment of boundary values for time-harmonic force:

$$\tilde{\Phi}(\mathbf{r}, \omega) = \int_V \tilde{G}(|\mathbf{r} - \mathbf{r}'|, \omega) \tilde{f}(\mathbf{r}', \omega) d^3 r' +$$

$$\oint_S (\tilde{\Phi}(\mathbf{r}', \omega) \nabla' \tilde{G}(|\mathbf{r} - \mathbf{r}'|, \omega) - \tilde{G}(|\mathbf{r} - \mathbf{r}'|, \omega) \nabla' \tilde{\Phi}(\mathbf{r}', \omega)) \cdot \hat{\mathbf{n}} d^2 r'$$

Boundary values for our example :

$$\left(\frac{\partial \tilde{\Phi}}{\partial z} \right)_{z=0} = \begin{cases} 0 & \text{for } x^2 + y^2 > a^2 \\ i\omega \varepsilon a & \text{for } x^2 + y^2 < a^2 \end{cases}$$

Note: Need Green's function with vanishing gradient at $z=0$:

$$\tilde{G}(|\mathbf{r} - \mathbf{r}'|, \omega) = \frac{e^{ik|\mathbf{r} - \mathbf{r}'|}}{4\pi|\mathbf{r} - \mathbf{r}'|} + \frac{e^{ik|\mathbf{r} - \bar{\mathbf{r}}'|}}{4\pi|\mathbf{r} - \bar{\mathbf{r}}'|} \quad \text{where } \bar{z}' = -z'; \quad z > 0$$

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$$\tilde{\Phi}(\mathbf{r}, \omega) = - \oint_{S: z'=0} \tilde{G}(|\mathbf{r} - \mathbf{r}'|, \omega) \frac{\partial \tilde{\Phi}(\mathbf{r}', \omega)}{\partial z'} dx' dy'$$

$$\tilde{G}(|\mathbf{r} - \mathbf{r}'|, \omega) = \frac{e^{ik|\mathbf{r} - \mathbf{r}'|}}{4\pi|\mathbf{r} - \mathbf{r}'|} + \frac{e^{ik|\mathbf{r} - \bar{\mathbf{r}}'|}}{4\pi|\mathbf{r} - \bar{\mathbf{r}}'|} \quad \text{where } \bar{z}' = -z'; \quad z > 0$$

$$\tilde{G}(|\mathbf{r} - \mathbf{r}'|, \omega)_{z'=0} = \frac{e^{ik|\mathbf{r} - \mathbf{r}'|}}{2\pi|\mathbf{r} - \mathbf{r}'|} \Big|_{z'=0}; \quad z > 0$$

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$$\begin{aligned}\tilde{\Phi}(\mathbf{r}, \omega) &= - \oint_{S: z'=0} \tilde{G}(|\mathbf{r}-\mathbf{r}'|, \omega) \frac{\partial \tilde{\Phi}(\mathbf{r}', \omega)}{\partial z'} dx' dy' \\ &= -i\omega\epsilon a \int_0^a r' dr' \int_0^{2\pi} d\phi' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{2\pi|\mathbf{r}-\mathbf{r}'|} \Big|_{z'=0}\end{aligned}$$

Integration domain : $x' = r' \cos \phi'$
 $y' = r' \sin \phi'$

For $r \gg a$; $|\mathbf{r}-\mathbf{r}'| \approx r - \hat{\mathbf{r}} \cdot \mathbf{r}'$

Assume $\hat{\mathbf{r}}$ is in the yz plane; $\phi = \frac{\pi}{2}$

$$\hat{\mathbf{r}} = \sin \theta \hat{\mathbf{y}} + \cos \theta \hat{\mathbf{z}}$$

$$|\mathbf{r}-\mathbf{r}'| \approx r - \hat{\mathbf{r}} \cdot \mathbf{r}' = r - r' \sin \theta \sin \phi'$$

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$$\tilde{\Phi}(\mathbf{r}, \omega) = -\frac{i\omega\epsilon a}{2\pi} \frac{e^{ikr}}{r} \int_0^a r' dr' \int_0^{2\pi} d\phi' e^{-ikr' \sin \theta \sin \phi'}$$

Note that : $\frac{1}{2\pi} \int_0^{2\pi} d\phi' e^{-iu \sin \phi'} = J_0(u)$

$$\Rightarrow \tilde{\Phi}(\mathbf{r}, \omega) = -i\omega\epsilon a \frac{e^{ikr}}{r} \int_0^a r' dr' J_0(kr' \sin \theta)$$

$$\int_0^w u du J_0(u) = w J_1(w)$$

$$\Rightarrow \tilde{\Phi}(\mathbf{r}, \omega) = -i\omega\epsilon a^3 \frac{e^{ikr}}{r} \frac{J_1(ka \sin \theta)}{ka \sin \theta}$$

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Energy flux : $\mathbf{j}_e = \delta \mathbf{v} p$

Taking time average : $\langle \mathbf{j}_e \rangle = \frac{1}{2} \Re(\delta \mathbf{v} p^*)$
 $= \frac{1}{2} \rho_0 \Re((- \nabla \Phi)(-i\omega \Phi)^*)$

Time averaged power per solid angle :

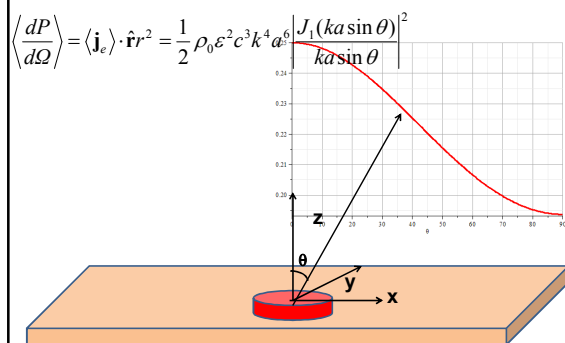
$$\left\langle \frac{dP}{d\Omega} \right\rangle = \langle \mathbf{j}_e \rangle \cdot \hat{\mathbf{r}} r^2 = \frac{1}{2} \rho_0 \epsilon^2 c^3 k^4 a^6 \left| \frac{J_1(ka \sin \theta)}{ka \sin \theta} \right|^2$$

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Time averaged power per solid angle :



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Scattering of sound waves –
for example, from a rigid cylinder

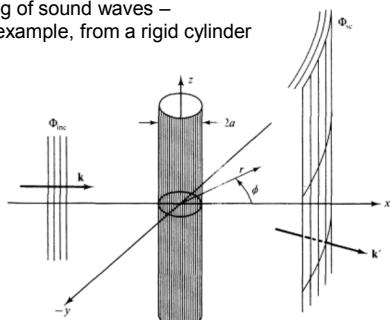


Figure 51.8 Scattering from a rigid cylinder.

Figure from Fetter and Walecka pg. 337

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Scattering of sound waves –
for example, from a rigid cylinder

Velocity potential --

$$\Phi(\mathbf{r}) = \Phi_{inc}(\mathbf{r}) + \Phi_{sc}(\mathbf{r}) \quad \Phi_{inc}(\mathbf{r}) = e^{ikr}$$

Helmholz equation in cylindrical coordinates:

$$(\nabla^2 + k^2)\Phi(\mathbf{r}) = 0 = \left(\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial}{\partial \phi^2} + \frac{\partial}{\partial z^2} + k^2 \right) \Phi(\mathbf{r})$$

$$\text{Assume: } \Phi(\mathbf{r}) = \sum_{m=-\infty}^{\infty} e^{im\phi} R_m(r)$$

$$\text{where } \left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{m^2}{r^2} + k^2 \right) R_m(r) = 0$$

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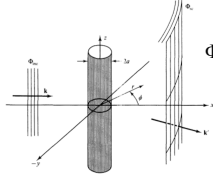


Figure 51.8 Scattering from a rigid cylinder.

$$\Phi_{inc}(\mathbf{r}) = e^{i\mathbf{k} \cdot \mathbf{r}} = e^{ikr \cos \phi} = \sum_{m=-\infty}^{\infty} i^m e^{im\phi} J_m(kr)$$

$\Phi_{sc}(\mathbf{r}) = \sum_{m=-\infty}^{\infty} C_m e^{im\phi} H_m(kr)$ where Hankel function represents an outgoing wave: $H_m(kr) = J_m(kr) + iN_m(kr)$

Boundary condition at $r = a$: $\left. \frac{\partial \Phi}{\partial r} \right|_{r=a} = 0$

$$\Rightarrow i^m J'_m(ka) + C_m H'_m(ka) = 0 \quad C_m = -i^m \frac{J'_m(ka)}{H'_m(ka)}$$

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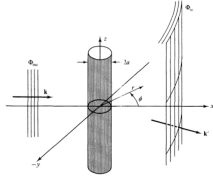


Figure 51.8 Scattering from a rigid cylinder.

$$\Phi_{sc}(\mathbf{r}) = - \sum_{m=-\infty}^{\infty} i^m \frac{J'_m(ka)}{H'_m(ka)} e^{im\phi} H_m(kr)$$

Asymptotic form:

$$i^m H_m(kr) \approx \sqrt{\frac{2}{\pi kr}} e^{i(kr - \pi/4)}$$

$$\Phi_{sc}(\mathbf{r}) \approx f(\phi) \sqrt{\frac{1}{r}} e^{ikr} = - \sum_{m=-\infty}^{\infty} \frac{J'_m(ka)}{H'_m(ka)} e^{im\phi} \sqrt{\frac{2}{\pi kr}} e^{i(kr - \pi/4)}$$

$$\Rightarrow f(\phi) = - \sqrt{\frac{2}{\pi k}} \sum_{m=-\infty}^{\infty} \frac{J'_m(ka)}{H'_m(ka)} e^{i(m\phi - \pi/4)}$$

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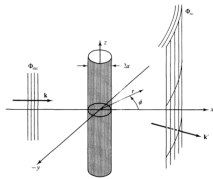
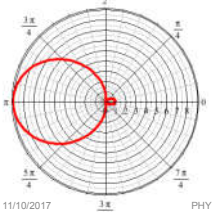


Figure 51.8 Scattering from a rigid cylinder.

$$\frac{d\sigma}{d\phi} = |f(\phi)|^2$$

$$f(\phi) = - \sqrt{\frac{2}{\pi k}} \sum_{m=-\infty}^{\infty} \frac{J'_m(ka)}{H'_m(ka)} e^{i(m\phi - \pi/4)}$$

For $ka \ll 1$

$$\frac{d\sigma}{d\phi} = |f(\phi)|^2 \approx \frac{1}{8} \pi k^3 a^4 (1 - 2 \cos \phi)^2$$


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