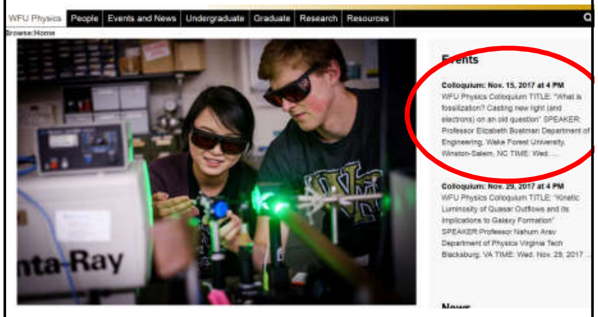
PHY 711 Classical Mechanics and Mathematical Methods
9-9:50 AM MWF Olin 107

Plan for Lecture 31: Chap. 9 of F&W

Wave equation for sound in the linear approximation

1. Sound scattering
2. Non-linear effects in sound waves

22	Mon, 10/23/2017	Chap. 7	Solutions of Sturm-Liouville equations		
23	Wed, 10/25/2017	Chap. 7	Solutions of Sturm-Liouville equations		
24	Fri, 10/27/2017	App. A	Laplace transforms and contour integrals	#13	11/01/2017
25	Mon, 10/30/2017	App. A	Contour integrals		
26	Wed, 11/01/2017	Chap. 8	Mechanics of Elastic Membranes	#14	11/06/2017
27	Fri, 11/03/2017	Chap. 9	Introduction to hydrodynamics		
28	Mon, 11/06/2017	Chap. 9	Introduction to hydrodynamics	#15	11/10/2017
29	Wed, 11/08/2017	Chap. 9	Sound waves		
30	Fri, 11/10/2017	Chap. 9	Sound waves	#16	11/17/2017
	Mon, 11/13/2017		*Class cancelled*		
31	Wed, 11/15/2017	Chap. 9	Sound waves -- including non-linearities		
32	Fri, 11/17/2017	Chap. 10	Surface waves in fluids		
33	Mon, 11/20/2017	Chap. 10	Surface waves in fluids		
	Wed, 11/22/2017		*Thanksgiving Holiday -- No class*		
	Fri, 11/24/2017		*Thanksgiving Holiday -- No class*		
34	Mon, 11/27/2017	Chap. 11	Heat conductivity		
35	Wed, 11/29/2017	Chap. 12	Viscous fluids		
36	Fri, 12/01/2017	Chap. 12	Viscous fluids		
	Mon, 12/04/2017		Presentations I		
	Wed, 12/06/2017		Presentations II		
	Fri, 12/08/2017		Presentations III		



Events

Colloquium: Nov. 15, 2017 at 4 PM
WFU Physics Colloquium TITLE: "What is fossilization? Casting new light (and electrons) on an old question" SPEAKER: Professor Elizabeth Bostman Department of Engineering, Wake Forest University, Winston-Salem, NC TIME: Wed. ...

Colloquium: Nov. 29, 2017 at 4 PM
WFU Physics Colloquium TITLE: "Kinetic Luminosity of Quasar Outflows and its Implications to Galaxy Formation" SPEAKER Professor Nahum Arav Department of Physics Virginia Tech Blacksburg, VA TIME: Wed. Nov. 29, 2017 ...

News

Solutions to wave equation:

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = 0$$

Plane wave solution :

$$\Phi(\mathbf{r},t) = Ae^{i\mathbf{k}\cdot\mathbf{r}-i\omega t} \quad \text{where} \quad k^2 = \left(\frac{\omega}{c}\right)^2$$

Scattering of sound waves –
for example, from a rigid cylinder

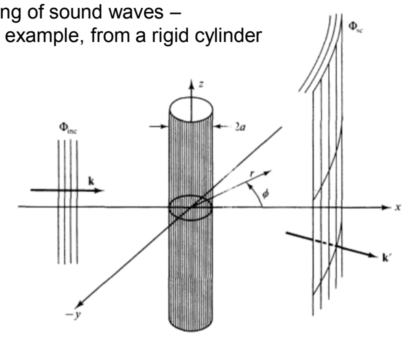


Figure 51.8 Scattering from a rigid cylinder.

Figure from Fetter and Walecka pg. 337

Scattering of sound waves –
for example, from a rigid cylinder

Velocity potential --

$$\Phi(\mathbf{r}) = \Phi_{inc}(\mathbf{r}) + \Phi_{sc}(\mathbf{r}) \qquad \Phi_{inc}(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}}$$

Helmholz equation in cylindrical coordinates:

$$\left(\nabla^2 + k^2\right)\Phi(\mathbf{r}) = 0 = \left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial\phi^2} + \frac{\partial}{\partial z^2} + k^2\right)\Phi(\mathbf{r})$$

Assume: $\Phi(\mathbf{r}) = \sum_{m=-\infty}^{\infty} e^{im\phi}R_m(r)$

where $\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m^2}{r^2} + k^2\right)R_m(r) = 0$

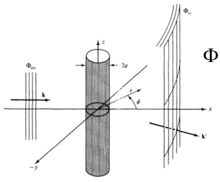


Figure 51.8 Scattering from a rigid cylinder.

$$\Phi_{inc}(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}} = e^{ikr\cos\phi} = \sum_{m=-\infty}^{\infty} i^m e^{im\phi} J_m(kr)$$

Note that for integer m, $J_m(z) = (-1)^m J_{-m}(z)$ so that this expression can be simplified.

$\Phi_{sc}(\mathbf{r}) = \sum_{m=-\infty}^{\infty} C_m e^{im\phi} H_m(kr)$ where Hankel function represents an outgoing wave: $H_m(kr) = J_m(kr) + iN_m(kr)$

Boundary condition at $r = a$: $\left.\frac{\partial\Phi}{\partial r}\right|_{r=a} = 0$

$$\Rightarrow i^m J'_m(ka) + C_m H'_m(ka) = 0 \qquad C_m = -i^m \frac{J'_m(ka)}{H'_m(ka)}$$

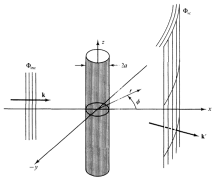


Figure 51.8 Scattering from a rigid cylinder.

$$\Phi_{sc}(\mathbf{r}) = -\sum_{m=-\infty}^{\infty} i^m \frac{J'_m(ka)}{H'_m(ka)} e^{im\phi} H_m(kr)$$

Asymptotic form:

$$i^m H_m(kr) \underset{kr\to\infty}{\approx} \sqrt{\frac{2}{\pi kr}} e^{i(kr-\pi/4)}$$

$$\Phi_{sc}(\mathbf{r}) \underset{kr\to\infty}{\approx} f(\phi)\sqrt{\frac{1}{r}} e^{ikr} = -\sum_{m=-\infty}^{\infty} \frac{J'_m(ka)}{H'_m(ka)} e^{im\phi} \sqrt{\frac{2}{\pi kr}} e^{i(kr-\pi/4)}$$

$$\Rightarrow f(\phi) = -\sqrt{\frac{2}{\pi k}} \sum_{m=-\infty}^{\infty} \frac{J'_m(ka)}{H'_m(ka)} e^{i(m\phi-\pi/4)}$$

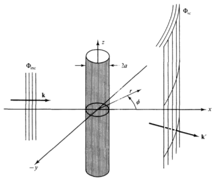
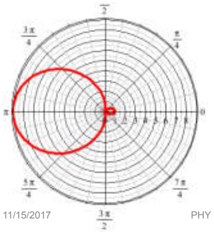


Figure 51.8 Scattering from a rigid cylinder.



$$\frac{d\sigma}{d\phi} = |f(\phi)|^2$$

$$f(\phi) = -\sqrt{\frac{2}{\pi k}} \sum_{m=-\infty}^{\infty} \frac{J'_m(ka)}{H'_m(ka)} e^{i(m\phi-\pi/4)}$$

For *ka << 1*

$$\frac{d\sigma}{d\phi} = |f(\phi)|^2 \approx \frac{1}{8}\pi k^3 a^4 (1 - 2\cos\phi)^2$$

Effects of nonlinearities in fluid equations
-- one dimensional case

Newton - Euler equation of motion :

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{f}_{applied} - \frac{\nabla p}{\rho}$$

Continuity equation : $\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0$

Assume spatial variation confined to x direction ;
assume that $\mathbf{v} = v\hat{\mathbf{x}}$ and $\mathbf{f}_{applied} = 0$.

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x} + \rho\frac{\partial v}{\partial x} = 0$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0$$

$$\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x} + \rho\frac{\partial v}{\partial x} = 0$$

Expressing p in terms of ρ: $p = p(\rho)$

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial \rho}\frac{\partial \rho}{\partial x} \equiv c^2(\rho)\frac{\partial \rho}{\partial x} \qquad \text{where } \frac{\partial p}{\partial \rho} \equiv c^2(\rho)$$

For adiabatic ideal gas: $\frac{dp}{p} = \gamma\frac{d\rho}{\rho}$ $\quad p = p_0\left(\frac{\rho}{\rho_0}\right)^{\gamma}$

$$c^2(\rho) = \frac{\gamma p}{\rho} = c_0^2\left(\frac{\rho}{\rho_0}\right)^{\gamma-1} \qquad \text{where } c_0^2 \equiv \frac{\gamma p_0}{\rho_0}$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{c^2(\rho)}{\rho}\frac{\partial \rho}{\partial x} = 0$$

$$\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x} + \rho\frac{\partial v}{\partial x} = 0$$

Expressing variation of v in terms of $v(\rho)$:

$$\frac{\partial v}{\partial \rho}\frac{\partial \rho}{\partial t} + v\frac{\partial v}{\partial \rho}\frac{\partial \rho}{\partial x} + \frac{c^2(\rho)}{\rho}\frac{\partial \rho}{\partial x} = 0$$

$$\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x} + \rho\frac{\partial v}{\partial \rho}\frac{\partial \rho}{\partial x} = 0$$

Some more algebra :

From Euler equation : $\frac{\partial v}{\partial \rho}\left(\frac{\partial \rho}{\partial t}+v\frac{\partial \rho}{\partial x}\right)+\frac{c^2(\rho)}{\rho}\frac{\partial \rho}{\partial x}=0$

From continuity equation : $\frac{\partial \rho}{\partial t}+v\frac{\partial \rho}{\partial x}=-\rho\frac{\partial v}{\partial \rho}\frac{\partial \rho}{\partial x}$

Combined equation : $\frac{\partial v}{\partial \rho}\left(-\rho\frac{\partial v}{\partial \rho}\frac{\partial \rho}{\partial x}\right)+\frac{c^2(\rho)}{\rho}\frac{\partial \rho}{\partial x}=0$

$$\Rightarrow\left(\frac{\partial v}{\partial \rho}\right)^2=\frac{c^2(\rho)}{\rho^2} \qquad \frac{\partial v}{\partial \rho}=\pm\frac{c}{\rho}$$

$$\Rightarrow\frac{\partial \rho}{\partial t}+(v\pm c)\frac{\partial \rho}{\partial x}=0$$

Assuming adiabatic process : $c^2=c_0^2\left(\frac{\rho}{\rho_0}\right)^{\gamma-1}$ $\qquad c_0^2=\frac{\gamma p_0}{\rho_0}$

$$\frac{\partial v}{\partial \rho}=\frac{dv}{d\rho}=\pm\frac{c}{\rho} \qquad \Rightarrow v=\pm c_0\int_{\rho_0}^{\rho}\left(\frac{\rho'}{\rho_0}\right)^{(\gamma-1)/2}\frac{d\rho'}{\rho'}$$

$$\Rightarrow v=\pm\frac{2c_0}{\gamma-1}\left(\left(\frac{\rho}{\rho_0}\right)^{(\gamma-1)/2}-1\right)$$

$$\Rightarrow c=c_0\left(\frac{\rho}{\rho_0}\right)^{(\gamma-1)/2}$$

Summary :

$$\frac{dv}{d\rho}=\pm\frac{c}{\rho}$$

$$\frac{\partial \rho}{\partial t}+(v\pm c)\frac{\partial \rho}{\partial x}=0$$

Assuming adiabatic process : $c^2=c_0^2\left(\frac{\rho}{\rho_0}\right)^{\gamma-1}$ $\qquad c_0^2=\frac{\gamma p_0}{\rho_0}$

$$c=c_0\left(\frac{\rho}{\rho_0}\right)^{(\gamma-1)/2} \qquad v=\pm\frac{2c_0}{\gamma-1}\left(\left(\frac{\rho}{\rho_0}\right)^{(\gamma-1)/2}-1\right)$$

Traveling wave solution:

Assume: $\rho = \rho_0 + f(x - u(\rho)t)$

Need to find self - consistent equations for propagation velocity $u(\rho)$ using equations

From previous derivations: $\frac{\partial \rho}{\partial t} + (v \pm c)\frac{\partial \rho}{\partial x} = 0$

Apparently: $u(\rho) \Leftrightarrow v \pm c$

For adiabatic ideal gas and + signs:

$$u = v + c = c_0\left(\frac{\gamma+1}{\gamma-1}\left(\frac{\rho}{\rho_0}\right)^{(\gamma-1)/2} - \frac{2}{\gamma-1}\right)$$

Traveling wave solution -- continued:

$$\frac{\partial \rho}{\partial t} + (v \pm c)\frac{\partial \rho}{\partial x} = 0$$

Assume: $\rho = \rho_0 + f(x - u(\rho)t) = \rho_0 + f(x - (v \pm c)t)$

For adiabatic ideal gas and + signs:

$$u = v + c = c_0\left(\frac{\gamma+1}{\gamma-1}\left(\frac{\rho}{\rho_0}\right)^{(\gamma-1)/2} - \frac{2}{\gamma-1}\right)$$

Solutionin linearapproxiation:

$$u = v + c \approx v_0 + c_0 = c_0\left(\frac{\gamma+1}{\gamma-1} - \frac{2}{\gamma-1}\right) = c_0$$

$$\Rightarrow \rho = \rho_0 + f(x - c_0 t)$$

Traveling wave solution -- full non-linear case:

Visualization for particular waveform: $\rho = \rho_0 + f(\underbrace{x - u(\rho)t}_{w})$

Assume: $f(w) \equiv \rho_0 s(w)$

$$\frac{\rho}{\rho_0} = 1 + s(x - ut)$$

For adiabatic ideal gas:

$$u = c_0\left(\frac{\gamma+1}{\gamma-1}\left(\frac{\rho}{\rho_0}\right)^{(\gamma-1)/2} - \frac{2}{\gamma-1}\right)$$

$$u = c_0\left(\frac{\gamma+1}{\gamma-1}\left(1 + s(x - ut)\right)^{(\gamma-1)/2} - \frac{2}{\gamma-1}\right)$$

Visualization continued:

$$u = c_0\left(\frac{\gamma+1}{\gamma-1}\big(1+s(x-ut)\big)^{(\gamma-1)/2} - \frac{2}{\gamma-1}\right)$$

Plot $s(x-ut)$ for fixed t, as a function of x:

Let $w = x - ut$

$$x = w + ut = w + u(w)t \equiv x(w,t)$$

$$u(w) = c_0\left(\frac{\gamma+1}{\gamma-1}\big(1+s(w)\big)^{(\gamma-1)/2} - \frac{2}{\gamma-1}\right)$$

Parametric equations:

plot $s(w)$ vs $x(w,t)$ for range of w at each t

Summary

$$\frac{\partial\rho}{\partial t} + u(\rho)\frac{\partial\rho}{\partial x} = 0$$

Solution: $\rho = \rho_0 + f(x-u(\rho)t) = \rho_0\big(1+s(x-u(\rho)t)\big)$

For linear case: $u(\rho) = c_0$

For non-linear case: $u(\rho) = c_0\left(\frac{\gamma+1}{\gamma-1}\big(1+s(x-ut)\big)^{(\gamma-1)/2} - \frac{2}{\gamma-1}\right)$

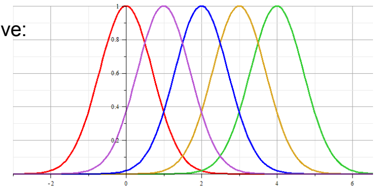
Plot $s(x-ut)$ for fixed t, as a function of x:

Let $w = x - ut \;\;\Rightarrow\;\; x = w + ut = w + u(w)t \equiv x(w,t)$

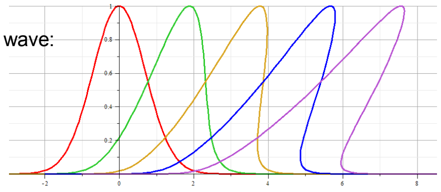
$$u(w) = c_0\left(\frac{\gamma+1}{\gamma-1}\big(1+s(w)\big)^{(\gamma-1)/2} - \frac{2}{\gamma-1}\right)$$

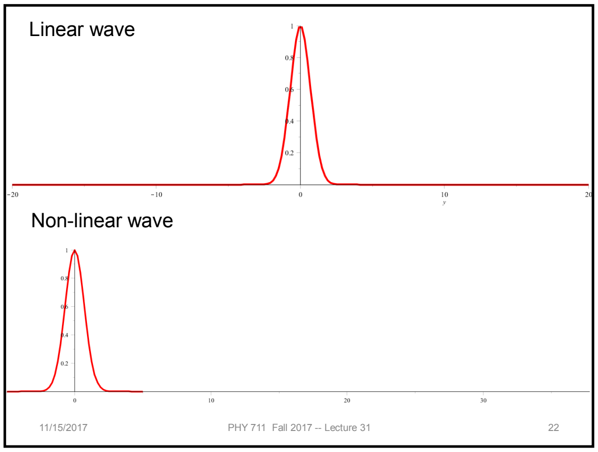
Parametric equations: plot $s(w)$ vs $x(w,t)$ for range of w

Linear wave:



Non-linear wave:

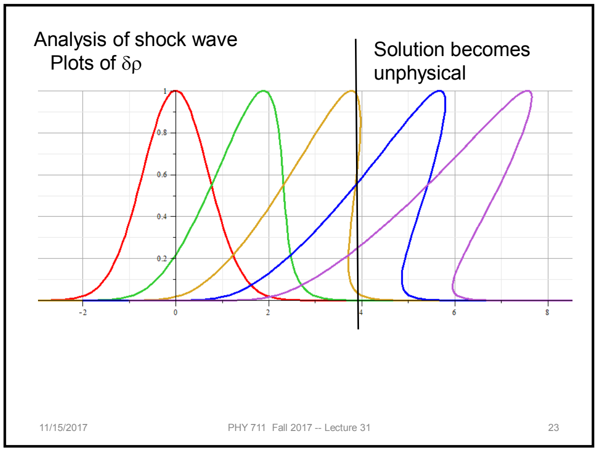




Linear wave

Non-linear wave

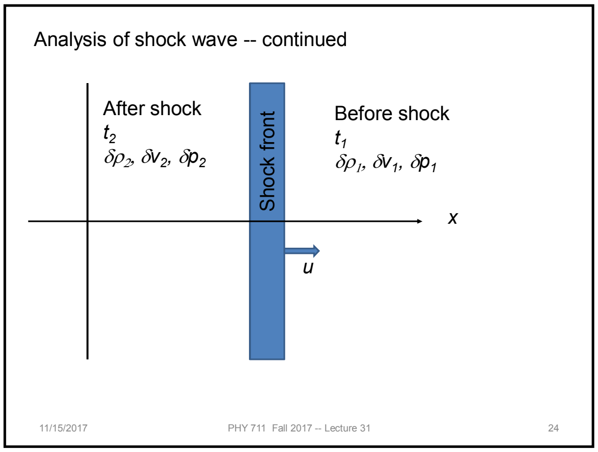
11/15/2017 PHY 711 Fall 2017 -- Lecture 31 22



Analysis of shock wave
Plots of $\delta\rho$

Solution becomes unphysical

11/15/2017 PHY 711 Fall 2017 -- Lecture 31 23



Analysis of shock wave -- continued

After shock
t_2
$\delta\rho_2$, δv_2, δp_2

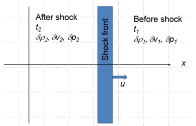
Shock front

Before shock
t_1
$\delta\rho_1$, δv_1, δp_1

11/15/2017 PHY 711 Fall 2017 -- Lecture 31 24

Analysis of shock wave – continued

While analysis in the shock region is complicated, we can use conservation laws to analyze regions 1 and 2



Assume $\rho(x,t) = \rho(x-ut)$

$p(x,t) = p(x-ut)$

$v(x,t) = v(x-ut)$

Continuity equation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} = 0 = \frac{\partial(\rho v - \rho u)}{\partial x} \quad \Rightarrow (v_2 - u)\rho_2 = (v_1 - u)\rho_1$$

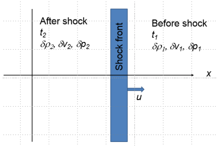
Conservation of energy and momentum:

$$\Rightarrow p_2 + \rho_2 (v_2 - u)^2 = p_1 + \rho_1 (v_1 - u)^2$$

$$\Rightarrow \epsilon_2 + \frac{1}{2}(v_2 - u)^2 + \frac{p_2}{\rho_2} = \epsilon_1 + \frac{1}{2}(v_1 - u)^2 + \frac{p_1}{\rho_1}$$

Analysis of shock wave – continued

For adiabatic ideal gas, also considering energy and momentum conservation:



$$\frac{\rho_2}{\rho_1} = \frac{\frac{\gamma+1}{\gamma-1}\frac{p_2}{p_1} + 1}{\frac{\gamma+1}{\gamma-1} + \frac{p_2}{p_1}} \leq \frac{\gamma+1}{\gamma-1}$$

Velocity relationships:

$$\frac{(v_1-u)^2}{c_1^2} = \frac{1}{2\gamma}\left(\gamma - 1 + (\gamma+1)\frac{p_2}{p_1}\right) \qquad \frac{(v_2-u)^2}{c_2^2} = \frac{1}{2\gamma}\left(\gamma - 1 + (\gamma+1)\frac{p_1}{p_2}\right)$$

where $c_1^2 \equiv \frac{\gamma p_1}{\rho_1}$ and $c_2^2 \equiv \frac{\gamma p_2}{\rho_2}$

For a strong shock:

$$\frac{(v_1-u)^2}{c_1^2} \to \frac{(\gamma+1)}{2\gamma}\frac{p_2}{p_1} \qquad \frac{(v_2-u)^2}{c_2^2} \to \frac{(\gamma-1)}{2\gamma}$$

Analysis of shock wave – continued

For adiabatic ideal gas, entropy considerations::

Internal energy density: $\varepsilon = \frac{p}{(\gamma-1)\rho} = C_V T$

First law of thermo: $d\varepsilon = Tds - pd\left(\frac{1}{\rho}\right)$

$$ds = \frac{1}{T}\left(d\left(\frac{p}{(\gamma-1)\rho}\right) - pd\left(\frac{1}{\rho}\right)\right) = C_V d\ln\left(\frac{p}{\rho^\gamma}\right)$$

$$s = C_V \ln\left(\frac{p}{\rho^\gamma}\right) + (\text{constant})$$

$$s_2 - s_1 = C_V \ln\left(\frac{p_2}{p_1}\left(\frac{\rho_1}{\rho_2}\right)^\gamma\right) \qquad 0 < s_2 - s_1 < C_V\left(\ln\left(\frac{p_2}{p_1}\right) - \gamma\ln\left(\frac{\gamma+1}{\gamma-1}\right)\right)$$