PHY 711 Classical Mechanics and Mathematical Methods
9-9:50 AM MWF Olin 107

Plan for Lecture 32:

Chapter 10 in F & W: Surface waves

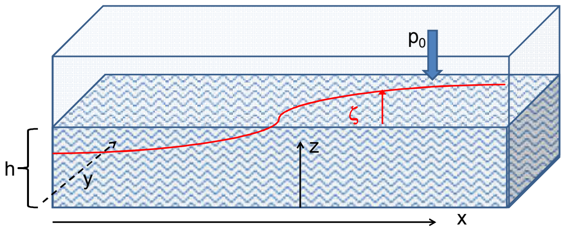
1. **Water waves in a channel**
2. **Wave-like solutions; wave speed**

23	Wed, 10/25/2017	Chap. 7	Solutions of Sturm-Liouville equations		
24	Fri, 10/27/2017	App. A	Laplace transforms and contour integrals	#13	11/01/2017
25	Mon, 10/30/2017	App. A	Contour integrals		
26	Wed, 11/01/2017	Chap. 8	Mechanics of Elastic Membranes	#14	11/06/2017
27	Fri, 11/03/2017	Chap. 9	Introduction to hydrodynamics		
28	Mon, 11/06/2017	Chap. 9	Introduction to hydrodynamics	#15	11/10/2017
29	Wed, 11/08/2017	Chap. 9	Sound waves		
30	Fri, 11/10/2017	Chap. 9	Sound waves	#16	11/17/2017
	Mon, 11/13/2017		*Class cancelled*		
31	Wed, 11/15/2017	Chap. 9	Sound waves -- including non-linearities		
32	Fri, 11/17/2017	Chap. 10	Surface waves in fluids	#17	11/27/2017
33	Mon, 11/20/2017	Chap. 10	Surface waves in fluids		
	Wed, 11/22/2017		*Thanksgiving Holiday -- No class*		
	Fri, 11/24/2017		*Thanksgiving Holiday -- No class*		
34	Mon, 11/27/2017	Chap. 11	Heat conductivity		
35	Wed, 11/29/2017	Chap. 12	Viscous fluids		
36	Fri, 12/01/2017	Chap. 12	Viscous fluids		
	Mon, 12/04/2017		Presentations I		
	Wed, 12/06/2017		Presentations II		
	Fri, 12/08/2017		Presentations III		

Physics of incompressible fluids and their surfaces

Reference: Chapter 10 of Fetter and Walecka

Consider a container of water with average height h and surface h+ζ(x,y,t); (h $\leftrightarrow$ z_0 on some of the slides)



Euler's equation for incompressible fluid:

$$\frac{d\mathbf{v}}{dt} = f_{applied} - \frac{\nabla p}{\rho} = -g\hat{\mathbf{z}} - \frac{\nabla p}{\rho}$$

Assume that $v_z << v_x, v_y$ $\quad \Rightarrow -g - \frac{1}{\rho}\frac{\partial p}{\partial z} \approx 0$

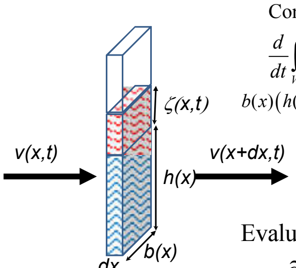
$$\Rightarrow p(x,y,z,t) = p_0 + \rho g\big(\zeta(x,y,t) + h - z\big)$$

Horizontal fluid motions (keeping leading terms):

$$\frac{dv_x}{dt} \approx \frac{\partial v_x}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} = -g\frac{\partial \zeta}{\partial x}$$

$$\frac{dv_y}{dt} \approx \frac{\partial v_y}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial y} = -g\frac{\partial \zeta}{\partial y}$$

Consider a surface *ζ(x,t)* wave moving in the *x*-direction in a channel of width *b(x)* and height *h(x):*



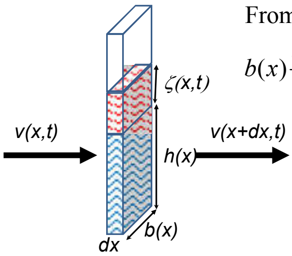
Continuity condition in integral form:

$$\frac{d}{dt}\int_V \rho dV + \int_A \rho \mathbf{v}\cdot d\mathbf{A} = 0$$

$$b(x)\big(h(x)+\zeta(x,t)\big)dx \qquad b(x)\big(h(x)+\zeta(x,t)\big)\hat{\mathbf{x}}$$

Evaluating continuity condition:

$$b(x)\frac{\partial \zeta}{\partial t} = -\frac{\partial}{\partial x}\big(h(x)b(x)v(x,t)\big)$$



From continuity condition:

$$b(x)\frac{\partial \zeta}{\partial t} = -\frac{\partial}{\partial x}\big(h(x)b(x)v(x,t)\big)$$

Example (Problem 10.3):

$$b(x) = b_0 \qquad h(x) = \kappa x$$

$$b_0\frac{\partial \zeta}{\partial t} = -\frac{\partial}{\partial x}\big((\kappa x)b_0 v(x,t)\big)$$

$$\frac{\partial \zeta}{\partial t} = -\kappa\left(v + x\frac{\partial v}{\partial x}\right)$$

From Newton-Euler equation:

$$\frac{dv}{dt} \approx \frac{\partial v}{\partial t} = -g\frac{\partial \zeta}{\partial x}$$

Example continued

$$\frac{\partial \zeta}{\partial t} = -\kappa\left(v + x\frac{\partial v}{\partial x}\right) \quad \Rightarrow \quad \frac{\partial^2 \zeta}{\partial t^2} = -\kappa\left(\frac{\partial v}{\partial t} + x\frac{\partial^2 v}{\partial x \partial t}\right)$$

$$\frac{\partial v}{\partial t} = -g\frac{\partial \zeta}{\partial x} \quad \Rightarrow \quad \frac{\partial^2 \zeta}{\partial t^2} = \kappa g\left(\frac{\partial \zeta}{\partial x} + x\frac{\partial^2 \zeta}{\partial x^2}\right)$$

It can be shown that a solution can take the form:

$$\zeta(x,t) = CJ_0\left(\frac{2\omega}{\sqrt{\kappa g}}\sqrt{x}\right)\cos(\omega t)$$

Note that $J_0(u)$ satisfies the equation: $\left(\frac{d^2}{du^2} + \frac{1}{u}\frac{d}{du} + 1\right)J_0(u) = 0$

Therefore, for $u = \frac{2\omega}{\sqrt{\kappa g}}\sqrt{x}$

$$\left(x\frac{d^2}{dx^2} + \frac{d}{dx}\right)J_0(u) = \frac{\omega^2}{\kappa g}\left(\frac{d^2}{du^2} + \frac{1}{u}\frac{d}{du}\right)J_0(u) = -\frac{\omega^2}{\kappa g}J_0(u)$$

Example continued

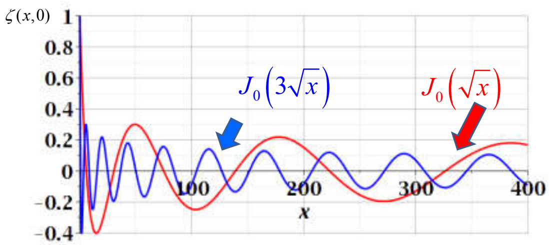
$$\frac{\partial^2 \zeta}{\partial t^2} = \kappa g\left(\frac{\partial \zeta}{\partial x} + x\frac{\partial^2 \zeta}{\partial x^2}\right)$$

$$\Rightarrow \zeta(x,t) = CJ_0\left(\frac{2\omega\sqrt{x}}{\sqrt{\kappa g}}\right)\cos(\omega t)$$

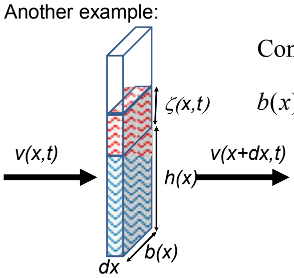
Check:

$$-\omega^2 CJ_0\left(\frac{2\omega\sqrt{x}}{\sqrt{\kappa g}}\right)\cos(\omega t) = \kappa g\left(\frac{\partial}{\partial x} + x\frac{\partial^2}{\partial x^2}\right)CJ_0\left(\frac{2\omega\sqrt{x}}{\sqrt{\kappa g}}\right)\cos(\omega t)$$

$$\zeta(x,t) = CJ_0\left(\frac{2\omega}{\sqrt{\kappa g}}\sqrt{x}\right)\cos(\omega t)$$



Another example:



Continuity condition:

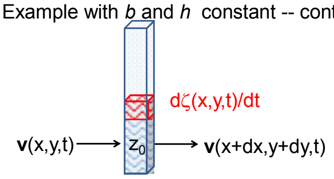
$$b(x)\frac{\partial \zeta}{\partial t} = -\frac{\partial}{\partial x}\big(h(x)b(x)v(x,t)\big)$$

Special case, where *b* and *h* are constant --

For constant b and h:

$$\frac{\partial \zeta}{\partial t} = -h\frac{\partial}{\partial x}\big(v(x,t)\big)$$

Example with *b* and *h* constant -- continued



Continuity condition for flow of incompressible fluid:

$$\frac{\partial \zeta}{\partial t} + h\nabla\cdot\mathbf{v} = 0$$

From horizontal flow relations: $\frac{\partial \mathbf{v}}{\partial t} = -g\nabla\zeta$

Equation for surface function: $\frac{\partial^2 \zeta}{\partial t^2} - gh\nabla^2\zeta = 0$

For uniform channel:

Surface wave equation:

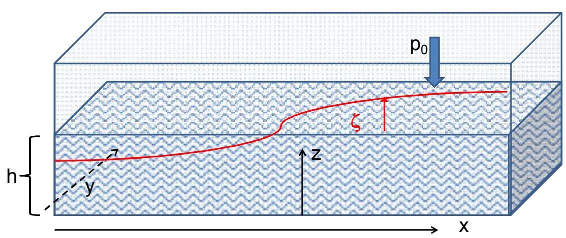
$$\frac{\partial^2 \zeta}{\partial t^2} - c^2 \nabla^2 \zeta = 0 \qquad c^2 = gh$$

More complete analysis finds:

$$c^2 = \frac{g}{k} \tanh(kh) \qquad \text{where } k = \frac{2\pi}{\lambda}$$

More details: -- recall setup --

Consider a container of water with average height h and surface h+ζ(x,y,t)



Equations describing fluid itself (without boundaries)

Euler's equation for incompressible fluid:

$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} = \frac{\partial \mathbf{v}}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) + \mathbf{v}\times\left(\nabla\times\mathbf{v}\right) = -\nabla U - \frac{\nabla p}{\rho}$$

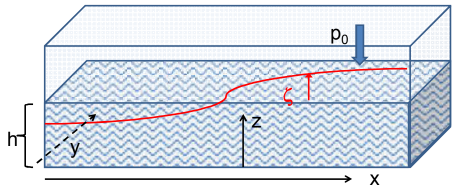
Assume that $\nabla\times\mathbf{v} = 0$ (irrotational flow) $\Rightarrow \mathbf{v} = -\nabla\Phi$

$$\Rightarrow \nabla\left(-\frac{\partial\Phi}{\partial t} + \tfrac{1}{2}v^2 + U + \frac{p}{\rho}\right) = 0$$

$$\Rightarrow -\frac{\partial\Phi}{\partial t} + \tfrac{1}{2}v^2 + U + \frac{p}{\rho} = \text{constant (within the fluid)}$$

For the same system, the continuity condition becomes

$$\nabla\cdot\mathbf{v} = -\nabla^2\Phi = 0$$



Within fluid: $0 \le z \le h+\zeta$

$$-\frac{\partial \Phi}{\partial t}+\tfrac{1}{2}v^2+g(z-h)=\text{constant}$$ (We have absorbed p_0 in "constant")

$$-\nabla^2\Phi=0$$

At surface: $z=h+\zeta$ with $\zeta=\zeta(x,y,t)$

$$\frac{d\zeta}{dt}=\frac{\partial \zeta}{\partial t}+v_x\frac{\partial \zeta}{\partial x}+v_y\frac{\partial \zeta}{\partial y} \quad \text{where } v_{x,y}=v_{x,y}(x,y,h+\zeta,t)$$

11/17/2017 PHY 711 Fall 2017 -- Lecture 32 16

Full equations:

Within fluid: $0 \le z \le h+\zeta$

$$-\frac{\partial \Phi}{\partial t}+\tfrac{1}{2}v^2+g(z-h)=\text{constant}$$ (We have absorbed p_0 in "constant")

$$-\nabla^2\Phi=0$$

At surface: $z=h+\zeta$ with $\zeta=\zeta(x,y,t)$

$$\frac{d\zeta}{dt}=\frac{\partial \zeta}{\partial t}+v_x\frac{\partial \zeta}{\partial x}+v_y\frac{\partial \zeta}{\partial y} \quad \text{where } v_{x,y}=v_{x,y}(x,y,h+\zeta,t)$$

Linearized equations:

$$\text{For} \quad 0\le z\le h+\zeta: \quad -\frac{\partial \Phi}{\partial t}+g(z-h)=0 \qquad -\nabla^2\Phi=0$$

$$\text{At surface}: \quad z=h+\zeta \qquad \frac{d\zeta}{dt}=\frac{\partial \zeta}{\partial t}=v_z(x,y,h+\zeta,t)$$

$$-\frac{\partial \Phi(x,y,h+\zeta,t)}{\partial t}+g\zeta=0$$

11/17/2017 PHY 711 Fall 2017 -- Lecture 32 17

For simplicity, keep only linear terms and assume that horizontal variation is only along *x*:

$$\text{For} \quad 0\le z\le h+\zeta: \qquad \nabla^2\Phi=\left(\frac{\partial^2}{\partial z^2}+\frac{\partial^2}{\partial x^2}\right)\Phi(x,z,t)=0$$

Consider and periodic waveform: $\Phi(x,z,t)=Z(z)\cos\big(k(x-ct)\big)$

$$\Rightarrow \left(\frac{d^2}{dz^2}-k^2\right)Z(z)=0$$

Boundary condition at bottom of tank: $v_z(x,0,t)=0$

$$\Rightarrow \frac{dZ}{dz}(0)=0 \qquad Z(z)=A\cosh(kz)$$

11/17/2017 PHY 711 Fall 2017 -- Lecture 32 18

For simplicity, keep only linear terms and assume that horizontal variation is only along *x* – continued:

$$\text{At surface}: \quad z = h+\zeta \qquad \frac{\partial \zeta}{\partial t} = v_z(x,h+\zeta,t) = -\frac{\partial \Phi(x,h+\zeta,t)}{\partial z}$$

$$-\frac{\partial \Phi(x,h+\zeta,t)}{\partial t} + g\zeta = 0$$

$$-\frac{\partial^2 \Phi(x,h+\zeta,t)}{\partial t^2} + g\frac{\partial \zeta}{\partial t} = -\frac{\partial^2 \Phi(x,h+\zeta,t)}{\partial t^2} - g\frac{\partial \Phi(x,h+\zeta,t)}{\partial z} = 0$$

$$\text{For} \quad \Phi(x,(h+\zeta),t) = A\cosh(k(h+\zeta))\cos(k(x-ct))$$

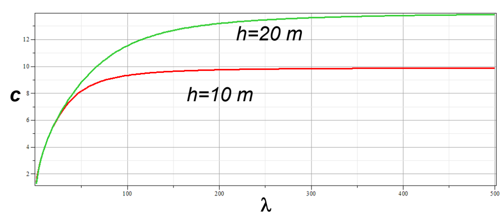
$$A\cosh(k(h+\zeta))\cos(k(x-ct))\left(k^2c^2 - gk\frac{\sinh(k(h+\zeta))}{\cosh(k(h+\zeta))}\right) = 0$$

$$\Rightarrow \mathrm{c}^2 = \frac{g}{k}\frac{\sinh(k(h+\zeta))}{\cosh(k(h+\zeta))}$$

For simplicity, keep only linear terms and assume that horizontal variation is only along *x* – continued:

$$c^2 = \frac{g}{k}\frac{\sinh(k(h+\zeta))}{\cosh(k(h+\zeta))} = \frac{g}{k}\tanh(k(h+\zeta))$$

$$\text{Assuming } \zeta << h: \qquad c^2 = \frac{g}{k}\tanh(kh) \qquad \lambda = \frac{2\pi}{k}$$



For simplicity, keep only linear terms and assume that horizontal variation is only along *x* – continued:

$$c^2 \approx \frac{g}{k}\tanh(kh) \qquad \text{For } \lambda >> h, \; c^2 \approx gh$$

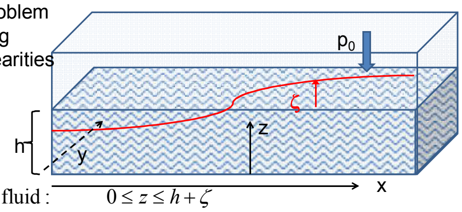
$$\Phi(x,z,t) = A\cosh(kz)\cos(k(x-ct))$$

$$\zeta(x,t) = \frac{1}{g}\frac{\partial \Phi(x,h+\zeta,t)}{\partial t} \approx \frac{kc}{g}A\cosh(kh)\sin(k(x-ct))$$

Note that for $\lambda >> h, \quad c^2 \approx gh$

(solutions are consistent with previous analysis)

General problem including non-linearities



Within fluid : $\quad 0 \le z \le h + \zeta$

$$-\frac{\partial \Phi}{\partial t} + \tfrac{1}{2} v^2 + g(z-h) = \text{constant} \quad \text{(We have absorbed}$$

$$-\nabla^2 \Phi = 0 \qquad\qquad p_0 \text{ in our constant.)}$$

At surface : $\quad z = h + \zeta \qquad$ with $\zeta = \zeta(x, y, t)$

$$\frac{d\zeta}{dt} = \frac{\partial \zeta}{\partial t} + v_x \frac{\partial \zeta}{\partial x} + v_y \frac{\partial \zeta}{\partial y} \quad \text{where } v_{x,y} = v_{x,y}(x, y, h+\zeta, t)$$

11/17/2017 PHY 711 Fall 2017 -- Lecture 32 22