PHY 711 Classical Mechanics and Mathematical Methods
9-9:50 AM MWF Olin 107

Plan for Lecture 33:

Chapter 10 in F & W: Surface waves

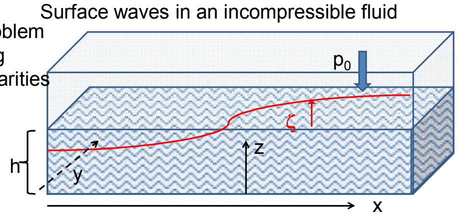
-- Non-linear contributions and soliton solutions

22	Mon, 10/23/2017	Chap. 7	Solutions of Sturm-Liouville equations		
23	Wed, 10/25/2017	Chap. 7	Solutions of Sturm-Liouville equations		
24	Fri, 10/27/2017	App. A	Laplace transforms and contour integrals	#13	11/01/2017
25	Mon, 10/30/2017	App. A	Contour integrals		
26	Wed, 11/01/2017	Chap. 8	Mechanics of Elastic Membranes	#14	11/06/2017
27	Fri, 11/03/2017	Chap. 9	Introduction to hydrodynamics		
28	Mon, 11/06/2017	Chap. 9	Introduction to hydrodynamics	#15	11/10/2017
29	Wed, 11/08/2017	Chap. 9	Sound waves		
30	Fri, 11/10/2017	Chap. 9	Sound waves	#16	11/17/2017
	Mon, 11/13/2017		*Class cancelled*		
31	Wed, 11/15/2017	Chap. 9	Sound waves -- including non-linearities		
32	Fri, 11/17/2017	Chap. 10	Surface waves in fluids	#17	11/27/2017
33	Mon, 11/20/2017	Chap. 10	Surface waves in fluids		
	Wed, 11/22/2017		*Thanksgiving Holiday -- No class*		
	Fri, 11/24/2017		*Thanksgiving Holiday -- No class*		
34	Mon, 11/27/2017	Chap. 11	Heat conductivity		
35	Wed, 11/29/2017	Chap. 12	Viscous fluids		
36	Fri, 12/01/2017	Chap. 12	Viscous fluids		
	Mon, 12/04/2017		Presentations I		
	Wed, 12/06/2017		Presentations II		
	Fri, 12/08/2017		Presentations III		

W

Surface waves in an incompressible fluid

General problem including non-linearities



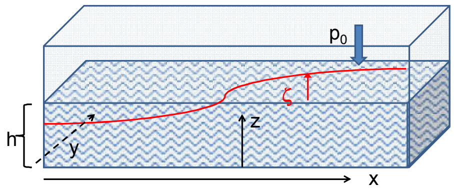
Within fluid: $0 \le z \le h+\zeta$

$$-\frac{\partial \Phi}{\partial t}+\tfrac{1}{2}v^2+g(z-h)=\text{constant} \qquad \Phi=\Phi(x,y,z,t)$$

$$-\nabla^2\Phi=0 \qquad \mathbf{v}=\mathbf{v}(x,y,z,t)=-\nabla\Phi(x,y,z,t)$$

At surface: $z=h+\zeta$ with $\zeta=\zeta(x,y,t)$

$$\frac{d\zeta}{dt}=\frac{\partial\zeta}{\partial t}+v_x\frac{\partial\zeta}{\partial x}+v_y\frac{\partial\zeta}{\partial y}=-\left.\frac{\partial\Phi(x,y,z,t)}{\partial z}\right|_{z=h+\zeta} \qquad \text{where } v_{x,y}=v_{x,y}(x,y,h+\zeta,t)$$



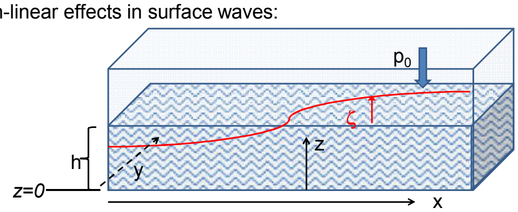
Further simplifications; assume trivial y - dependence

$\Phi = \Phi(x,z,t) \qquad \zeta = \zeta(x,t)$

Within fluid : $\quad 0 \le z \le h + \zeta$

At surface : $\quad v_z(x, z = h + \zeta, t) = -\dfrac{\partial \Phi}{\partial z} = \dfrac{d\zeta}{dt}$

Non-linear effects in surface waves:



Dominant non-linear effects $\Rightarrow$ soliton solutions

$$\zeta(x,t) = \eta_0 \operatorname{sech}^2\left(\sqrt{\frac{3\eta_0}{h}}\,\frac{x-ct}{2h}\right) \qquad \eta_0 = \text{constant}$$

$$\text{where } c = \sqrt{\frac{gh}{1-\eta_0/h}} \approx \sqrt{gh}\left(1+\frac{\eta_0}{2h}\right)$$

Detailed analysis of non-linear surface waves

[Note that these derivations follow Alexander L. Fetter and John Dirk Walecka, *Theoretical Mechanics of Particles and Continua* (McGraw Hill, 1980), Chapt. 10.]

We assume that we have an incompressible fluid: $\rho = \text{constant}$

Velocity potential: $\Phi(x,z,t)$; $\mathbf{v}(x,z,t) = -\nabla\Phi(x,z,t)$

The surface of the fluid is described by $z=h+\zeta(x,t)$. It is assumed that the fluid is contained in a structure (lake, river, swimming pool, etc.) with a structureless bottom defined by the $z = 0$ plane and filled to an equilibrium height of $z = h$.

Defining equations for $\Phi(x,z,t)$ and $\zeta(x,t)$

where $0 \le z \le h + \zeta(x,t)$

Continuity equation:

$$\nabla \cdot \mathbf{v} = 0 \quad \Rightarrow \quad \frac{\partial^2 \Phi(x,z,t)}{\partial x^2} + \frac{\partial^2 \Phi(x,z,t)}{\partial z^2} = 0$$

Bernoulli equation (assuming irrotational flow) and gravitation potential energy

$$-\frac{\partial \Phi(x,z,t)}{\partial t} + \frac{1}{2}\left[\underbrace{\left(\frac{\partial \Phi(x,z,t)}{\partial x}\right)^2}_{v_x^2} + \underbrace{\left(\frac{\partial \Phi(x,z,t)}{\partial z}\right)^2}_{v_z^2}\right] + g(z-h) = 0.$$

Boundary conditions on functions –

Zero velocity at bottom of tank:

$$\frac{\partial \Phi(x,0,t)}{\partial z} = 0.$$

Consistent vertical velocity at water surface

$$v_z(x,z,t)\rfloor_{z=h+\zeta} = \frac{d\zeta}{dt} = \mathbf{v} \cdot \nabla \zeta + \frac{\partial \zeta}{\partial t}$$

$$= v_x \frac{\partial \zeta}{\partial x} + \frac{\partial \zeta}{\partial t}$$

$$\Rightarrow -\frac{\partial \Phi(x,z,t)}{\partial z} + \frac{\partial \Phi(x,z,t)}{\partial x}\frac{\partial \zeta(x,t)}{\partial x} - \frac{\partial \zeta(x,t)}{\partial t}\rfloor_{z=h+\zeta} = 0$$

Analysis assuming water height *z* is small relative to variations in the direction of wave motion *(x)*
Taylor's expansion about *z* = 0:

$$\Phi(x,z,t) \approx \Phi(x,0,t) + z\frac{\partial \Phi}{\partial z}(x,0,t) + \frac{z^2}{2}\frac{\partial^2 \Phi}{\partial z^2}(x,0,t) + \frac{z^3}{3!}\frac{\partial^3 \Phi}{\partial z^3}(x,0,t) + \frac{z^4}{4!}\frac{\partial^4 \Phi}{\partial z^4}(x,0,t)\cdots$$

Note that the zero vertical velocity at the bottom ensures that all odd derivatives $\frac{\partial^n \Phi}{\partial z^n}(x,0,t)$ vanish from the

Taylor expansion . In addition, the Laplace equation allows us to convert all even derivatives with respect to *z* to derivatives with respect to *x*.

Modified Taylor's expansion: $\Phi(x,z,t) \approx \Phi(x,0,t) - \frac{z^2}{2}\frac{\partial^2 \Phi}{\partial x^2}(x,0,t) + \frac{z^4}{4!}\frac{\partial^4 \Phi}{\partial x^4}(x,0,t)\cdots$

Check linearized equations and their solutions:

Bernoulli equations --

Bernoulli equation evaluated at $z = h + \zeta(x,t)$

$$-\frac{\partial \Phi(x,h,t)}{\partial t} + g\zeta(x,t) = 0$$

Consistent vertical velocity at $z = h + \zeta(x,t)$

$$-\frac{\partial \Phi(x,z,t)}{\partial z} - \frac{\partial \zeta(x,t)}{\partial t}\Big]_{z=h+\zeta} = 0$$

Using Taylor's expansion results to lowest order

$$-\frac{\partial \Phi(x,h,t)}{\partial z} \approx h\frac{\partial^2 \Phi(x,0,t)}{\partial x^2} \qquad -\frac{\partial \Phi(x,h,t)}{\partial t} \approx -\frac{\partial \Phi(x,0,t)}{\partial t}$$

Decoupled equations: $\frac{\partial^2 \Phi(x,0,t)}{\partial t^2} = gh\frac{\partial^2 \Phi(x,0,t)}{\partial x^2}.$

Analysis of non-linear equations -- keeping the lowest order nonlinear terms and include up to 4th order derivatives in the linear terms. Let $\phi(x,t) \equiv \Phi(x,0,t)$

Approximate form of Bernoulli equation evaluated at surface: $z = h + \zeta$

$$-\frac{\partial \phi}{\partial t} + \frac{(h+\zeta)^2}{2}\frac{\partial^3 \phi}{\partial t \partial x^2} + \frac{1}{2}\left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left((h+\zeta)\frac{\partial^2 \phi}{\partial x^2}\right)^2\right] + g\zeta = 0$$

$$\Rightarrow -\frac{\partial \phi}{\partial t} + \frac{h^2}{2}\frac{\partial^3 \phi}{\partial t \partial x^2} + \frac{1}{2}\left(\frac{\partial \phi}{\partial x}\right)^2 + g\zeta = 0.$$

Approximate form of surface velocity expression :

$$\frac{\partial}{\partial x}\left((h+\zeta(x,t))\frac{\partial \phi}{\partial x}\right) - \frac{h^3}{3!}\frac{\partial^4 \phi}{\partial x^4} - \frac{\partial \zeta}{\partial t} = 0.$$

The expressions keep the lowest order nonlinear terms and include up to 4th order derivatives in the linear terms.

Coupled equations: $-\frac{\partial \phi}{\partial t} + \frac{h^2}{2}\frac{\partial^3 \phi}{\partial t \partial x^2} + \frac{1}{2}\left(\frac{\partial \phi}{\partial x}\right)^2 + g\zeta = 0.$

$$\frac{\partial}{\partial x}\left((h+\zeta(x,t))\frac{\partial \phi}{\partial x}\right) - \frac{h^3}{3!}\frac{\partial^4 \phi}{\partial x^4} - \frac{\partial \zeta}{\partial t} = 0.$$

Traveling wave solutions with new notation:

$u \equiv x - ct \qquad \phi(x,t) \equiv \chi(u) \quad$ and $\quad \zeta(x,t) \equiv \eta(u)$

Note that the wave "speed" *c* will be consistently determined

$$c\frac{d\chi(u)}{du} - \frac{ch^2}{2}\frac{d^3\chi(u)}{du^3} + \frac{1}{2}\left(\frac{d\chi(u)}{du}\right)^2 + g\eta(u) = 0.$$

$$\frac{d}{du}\left((h+\eta(u))\frac{d\chi(u)}{du}\right) - \frac{h^3}{6}\frac{d^4\chi(u)}{du^4} + c\frac{d\eta(u)}{du} = 0.$$

Integrating and re-arranging coupled equations

$$c\frac{d\chi(u)}{du}-\frac{ch^2}{2}\frac{d^3\chi(u)}{du^3}+\frac{1}{2}\left(\frac{d\chi(u)}{du}\right)^2+g\eta(u)=0.$$

$$\Rightarrow \chi'=-\frac{g}{c}\eta+\frac{h^2}{2}\chi'''-\frac{1}{2c}(\chi')^2\approx-\frac{g}{c}\eta-\frac{h^2g}{2c}\eta''-\frac{g^2}{2c^3}\eta^2$$

$$\frac{d}{du}\left((h+\eta(u))\frac{d\chi(u)}{du}\right)-\frac{h^3}{6}\frac{d^4\chi(u)}{du^4}+c\frac{d\eta(u)}{du}=0.$$

$$\Rightarrow (h+\eta)\chi'-\frac{h^3}{6}\chi'''+c\eta=0$$

Integrating and re-arranging coupled equations – continued --
Expressing modified surface velocity equation in terms of $\eta(u)$:

$$(h+\eta)\left(-\frac{g}{c}\eta-\frac{h^2g}{2c}\eta''-\frac{g^2}{2c^3}\eta^2\right)+\frac{h^3g}{6c}\eta''+c\eta=0$$

$$\Rightarrow\left(1-\frac{gh}{c^2}\right)\eta-\frac{gh^3}{3c^2}\eta''-\frac{g}{c^2}\left(1+\frac{gh}{2c^2}\right)\eta^2=0$$

$$\Rightarrow\left(1-\frac{hg}{c^2}\right)\eta(u)-\frac{h^2}{3}\eta''(u)-\frac{3}{2h}[\eta(u)]^2=0.$$

Note: $c^2=gh+\ldots$

Solution of the famous Korteweg-de Vries equation

Modified surface amplitude equation in terms of η

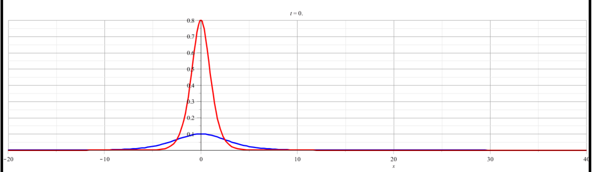
$$\Rightarrow\left(1-\frac{hg}{c^2}\right)\eta(u)-\frac{h^2}{3}\eta''(u)-\frac{3}{2h}[\eta(u)]^2=0.$$

Soliton solution

$$\zeta(x,t)=\eta(x-ct)=\eta_0\ \text{sech}^2\left(\sqrt{\frac{3\eta_0}{h}}\frac{x-ct}{2h}\right)$$

$$c=\sqrt{\frac{gh}{1-\eta_0/h}}\approx\sqrt{gh}\left(1+\frac{\eta_0}{2h}\right)\quad\text{where } \eta_0 \text{ is a constant}$$

$$\zeta(x,t) = \eta(x-ct) = \eta_0 \ \mathrm{sech}^2\left(\sqrt{\frac{3\eta_0}{h}}\ \frac{x-ct}{2h}\right)$$



Relationship to "standard" form of Korteweg-de Vries equation

New variables:

$$\beta = 2\eta_0, \quad \bar{x} = \sqrt{\frac{3}{2h}}\,\frac{x}{h}, \quad \text{and} \quad \bar{t} = \sqrt{\frac{3}{2h}}\,\frac{ct}{2\eta_0 h}.$$

Standard Korteweg-de Vries equation

$$\frac{\partial \eta}{\partial \bar{t}} + 6\eta\frac{\partial \eta}{\partial \bar{x}} + \frac{\partial^3 \eta}{\partial \bar{x}^3} = 0.$$

Soliton solution:

$$\eta(\bar{x},\bar{t}) = \frac{\beta}{2}\,\mathrm{sech}^2\left[\frac{\sqrt{\beta}}{2}(\bar{x}-\beta\bar{t})\right].$$

More details

Modified surface amplitude equation in terms of η :

$$\left(1-\frac{hg}{c^2}\right)\eta(u) - \frac{h^2}{3}\eta''(u) - \frac{3}{2h}\left[\eta(u)\right]^2 = 0.$$

Some identities: $\frac{\eta_0}{h} = 1-\frac{gh}{c^2}$; $\frac{\partial \eta}{\partial t} = -c\frac{d\eta}{du}$; $\frac{\partial \eta}{\partial x} = \frac{d\eta}{du}$.

Derivative of surface amplitude equation:

$$\frac{\eta_0}{h}\eta' - \frac{h^2}{3}\eta''' - \frac{3}{h}\eta\eta' = 0.$$

Expression in terms of x and t:

$$-\frac{\eta_0}{ch}\frac{\partial \eta}{\partial t} - \frac{h^2}{3}\frac{\partial^3 \eta}{\partial x^3} - \frac{3}{h}\eta\frac{\partial \eta}{\partial x} = 0.$$

Expression in terms of $\bar{x}$ and $\bar{t}$:

$$\frac{\partial \eta}{\partial \bar{t}} + 6\eta\frac{\partial \eta}{\partial \bar{x}} + \frac{\partial^3 \eta}{\partial \bar{x}^3} = 0.$$

Steps to solution

$$\left(1-\frac{hg}{c^2}\right)\eta(u)-\frac{h^2}{3}\eta''(u)-\frac{3}{2h}\left[\eta(u)\right]^2=0.$$

Let $1-\frac{hg}{c^2}\equiv\frac{\eta_0}{h}$ $\Rightarrow \frac{\eta_0}{h}\eta(u)-\frac{h^2}{3}\eta''(u)-\frac{3}{2h}\left[\eta(u)\right]^2=0.$

Multiply equation by $\eta'(u)$ $\Rightarrow \frac{d}{du}\left(\frac{\eta_0}{2h}\eta^2(u)-\frac{h^2}{6}\eta'^2(u)-\frac{1}{2h}\eta^3(u)\right)=0$

Integrate wrt u and assume solution vanishes for $u\rightarrow\infty$

$$\frac{\eta_0}{2h}\eta^2(u)-\frac{h^2}{6}\eta'^2(u)-\frac{1}{2h}\eta^3(u)=0$$

$$\eta'^2(u)=\frac{3}{h^3}\eta^2(u)\left(\eta_0-\eta(u)\right)$$

$$\frac{d\eta}{\eta\left(\eta_0-\eta\right)^{1/2}}=\sqrt{\frac{3}{h^3}}du \qquad \Rightarrow \eta(u)=\frac{\eta_0}{\cosh^2\left(\sqrt{\frac{3\eta_0}{4h^3}}u\right)}$$

Summary

Soliton solution

$$\zeta(x,t)=\eta(x-ct)=\eta_0\ \text{sech}^2\left(\sqrt{\frac{3\eta_0}{h}}\frac{x-ct}{2h}\right)$$

$$c=\sqrt{\frac{gh}{1-\eta_0/h}}\approx\sqrt{gh}\left(1+\frac{\eta_0}{2h}\right) \quad \text{where } \eta_0 \text{ is a constant}$$

Photo of canal soliton http://www.ma.hw.ac.uk/solitons/