PHY 711 Classical Mechanics and Mathematical Methods
9-9:50 AM MWF Olin 107

Plan for Lecture 34:

Chapter 11 in F & W:

Heat conduction

1. **Basic equations**
2. **Boundary value problems**

23	Wed, 10/25/2017	Chap. 7	Solutions of Sturm-Liouville equations		
24	Fri, 10/27/2017	App. A	Laplace transforms and contour integrals	#13	11/01/2017
25	Mon, 10/30/2017	App. A	Contour integrals		
26	Wed, 11/01/2017	Chap. 8	Mechanics of Elastic Membranes	#14	11/06/2017
27	Fri, 11/03/2017	Chap. 9	Introduction to hydrodynamics		
28	Mon, 11/06/2017	Chap. 9	Introduction to hydrodynamics	#15	11/10/2017
29	Wed, 11/08/2017	Chap. 9	Sound waves		
30	Fri, 11/10/2017	Chap. 9	Sound waves	#16	11/17/2017
	Mon, 11/13/2017		*Class cancelled*		
31	Wed, 11/15/2017	Chap. 9	Sound waves -- including non-linearities		
32	Fri, 11/17/2017	Chap. 10	Surface waves in fluids	#17	11/27/2017
33	Mon, 11/20/2017	Chap. 10	Surface waves in fluids		
	Wed, 11/22/2017		*Thanksgiving Holiday -- No class*		
	Fri, 11/24/2017		*Thanksgiving Holiday -- No class*		
34	Mon, 11/27/2017	Chap. 11	Heat conductivity		
35	Wed, 11/29/2017	Chap. 12	Viscous fluids		
36	Fri, 12/01/2017	Chap. 12	Viscous fluids		
	Mon, 12/04/2017		Presentations I		
	Wed, 12/06/2017		Presentations II		
	Fri, 12/08/2017		Presentations III		

Schedule for PHY 711 presentations

Monday Dec. 4, 2017

Time	Presenter	Title
9:00-9:15 AM	Ali Daraei	Rheology of fibrin fibers
9:17-9:32 AM		
9:34-9:49 A	Yan Li	Computational study of normal modes for one-dimensional mass-spring oscillating systems

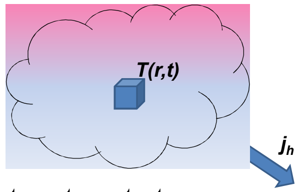
Wednesday Dec. 6, 2017

Time	Presenter	Title
9:00-9:15 AM	Matthew Waldrip	
9:17-9:32 AM	Nouf Alharbi	Vibrational Modes of Carbon Dioxide
9:34-9:49 AM	Kevin Roebuck	Equations of motion of the Foucault pendulum

Friday Dec. 8, 2017

Time	Presenter	Title
9:00-9:15 AM	Haardik Pandey	Normal modes of vibrations for water molecule
9:17-9:32 AM	Ellie Alipour	
9:34-9:49 AM	TJ Colvin	Quasiharmonic analysis of biomolecules

Conduction of heat



Enthalpy of a system at constant pressure p

non uniform temperature $T(\mathbf{r},t)$

mass density ρ and heat capacity c_p

$$H = \int_V \rho c_p \left(T(\mathbf{r},t) - T_0 \right) d^3r + H_0(T_0, p)$$

Note that in this treatment we are considering a system at constant pressure *p*

Notation:		
	Heat added to system	-- $dQ = TdS$
	External work done on system	-- $dW = -pdV$
	Internal energy	-- $dE = dQ + dW = TdS - pdV$
	Entropy	-- dS
	Enthalpy	-- $dH = d(E + pV) = TdS + Vdp$

Heat capacity at constant pressure:

$$C_p \equiv \left(\frac{dQ}{dT}\right)_p = \left(\frac{\partial H}{\partial T}\right)_p = T\left(\frac{\partial S}{\partial T}\right)_p$$

$$C_p = \int \rho c_p d^3r$$

More generally, note that c_p can depend on *T*; we are assume that dependence to be trivial.

Conduction of heat -- continued

$$H = \int_V \rho c_p \left(T(\mathbf{r},t) - T_0 \right) d^3r + H_0(T_0, p)$$

Time rate of change of enthapy:

$$\frac{dH}{dt} = \int_V \rho c_p \frac{\partial T(\mathbf{r},t)}{\partial t} d^3r = -\int_A \mathbf{j}_h \cdot d\mathbf{A} + \int_V \rho \dot{q} d^3r$$

heat flux

heat source

$$\rho c_p \frac{\partial T(\mathbf{r},t)}{\partial t} = -\nabla \cdot \mathbf{j}_h + \rho \dot{q}$$

Conduction of heat -- continued

$$\rho c_p \frac{\partial T(\mathbf{r},t)}{\partial t} = -\nabla \cdot \mathbf{j}_h + \rho \dot{q}$$

Empirically: $\mathbf{j}_h = -k_{th} \nabla T(\mathbf{r},t)$

$$\Rightarrow \frac{\partial T(\mathbf{r},t)}{\partial t} = \kappa \nabla^2 T(\mathbf{r},t) + \frac{\dot{q}}{c_p}$$

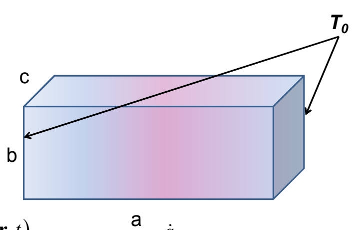
$\kappa \equiv \frac{k_{th}}{\rho c_p}$ thermal diffusivity

https://www.engineersedge.com/heat_transfer/thermal_diffusivity_table_13953.htm

Typical values (m^2/s)

Air	2×10^{-5}
Water	1×10^{-7}
Copper	1×10^{-4}

Boundary value problems for heat conduction

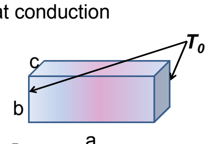


$$\frac{\partial T(\mathbf{r},t)}{\partial t} - \kappa \nabla^2 T(\mathbf{r},t) = \frac{\dot{q}}{c_p}$$

Without source term: $\frac{\partial T(\mathbf{r},t)}{\partial t} - \kappa \nabla^2 T(\mathbf{r},t) = 0$

Example with boundary values: $T(0,y,z,t) = T(a,y,z,t) = T_0$

Boundary value problems for heat conduction



$$\frac{\partial T(\mathbf{r},t)}{\partial t} - \kappa \nabla^2 T(\mathbf{r},t) = 0$$

$$T(0,y,z,t) = T(a,y,z,t) = T_0$$

$$\frac{\partial T(x,0,z,t)}{\partial y} = \frac{\partial T(x,b,z,t)}{\partial y} = 0$$

$$\frac{\partial T(x,y,0,t)}{\partial z} = \frac{\partial T(x,y,c,t)}{\partial z} = 0$$

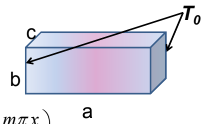
Assuming thermally insulated boundaries

Separation of variables: $T(x,y,z,t) = T_0 + X(x)Y(y)Z(z)e^{-\lambda t}$

Let $\frac{d^2X}{dx^2} = -\alpha^2 X \quad \frac{d^2Y}{dy^2} = -\beta^2 Y \quad \frac{d^2Z}{dz^2} = -\gamma^2 Z$

$$\Rightarrow -\lambda + \kappa(\alpha^2 + \beta^2 + \gamma^2) = 0$$

Boundary value problems for heat conduction



$$T(x,y,z,t) = T_0 + X(x)Y(y)Z(z)e^{-\lambda t}$$

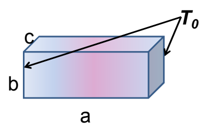
$$X(0) = X(a) = 0 \qquad \Rightarrow X(x) = \sin\left(\frac{m\pi x}{a}\right)$$

$$\frac{dY(0)}{dy} = \frac{dY(b)}{dy} = 0 \qquad \Rightarrow Y(y) = \cos\left(\frac{n\pi y}{b}\right)$$

$$\frac{dZ(0)}{dz} = \frac{dZ(c)}{dz} = 0 \qquad \Rightarrow Z(z) = \cos\left(\frac{p\pi z}{c}\right)$$

$$-\lambda_{nmp} + \kappa\left(\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 + \left(\frac{p\pi}{c}\right)^2\right) = 0$$

Boundary value problems for heat conduction



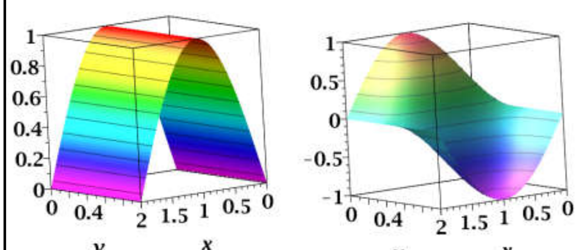
Full solution:

$$T(x,y,z,t) = T_0 + \sum_{nmp} C_{nmp} \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\cos\left(\frac{p\pi z}{c}\right)e^{-\lambda_{mnp} t}$$

$$\lambda_{nmp} = \kappa\left(\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 + \left(\frac{p\pi}{c}\right)^2\right)$$

Full solution:

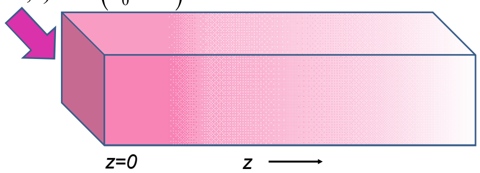
$$T(x,y,z,t) = T_0 + \sum_{nmp} C_{nmp} \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\cos\left(\frac{p\pi z}{c}\right)e^{-\lambda_{mnp} t}$$



$m = 1, n = 0, p = 0$ $\qquad$ $m = 1, m = 1, p = 0$

Oscillatory thermal behavior

$$T(z=0,t)=\Re\left(T_0 e^{-i\omega t}\right)$$



$$\frac{\partial T}{\partial t}=\kappa\frac{\partial^2 T}{\partial z^2}$$

Assume: $T(z,t)=\Re\left(f(z)e^{-i\omega t}\right)$

$$(-i\omega)f=\kappa\frac{d^2 f}{dz^2}$$

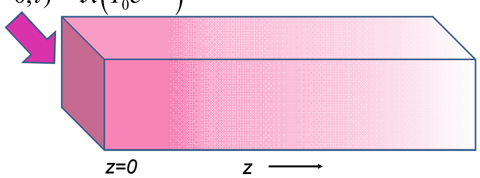
Let $f(z)=Ae^{\alpha z}$

$$\alpha^2=-\frac{i\omega}{\kappa}=e^{3i\pi/2}\ \frac{\omega}{\kappa}$$

$$\alpha=\pm(1-i)\sqrt{\frac{\omega}{2\kappa}}$$

Oscillatory thermal behavior -- continued

$$T(z=0,t)=\Re\left(T_0 e^{-i\omega t}\right)$$



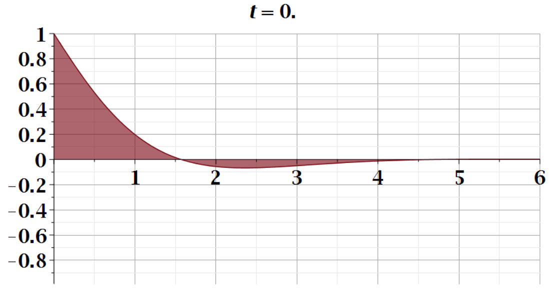
$$T(z,t)=\Re\left(Ae^{\pm(1-i)z/\delta}e^{-i\omega t}\right)$$

where $\delta\equiv\sqrt{\frac{2\kappa}{\omega}}$

Physical solution: $T(z,t)=T_0 e^{-z/\delta}\cos\left(\frac{z}{\delta}-\omega t\right)$

$$T(z,t)=T_0 e^{-z/\delta}\cos\left(\frac{z}{\delta}-\omega t\right)$$

$t = 0.$



Initial value problem in an infinite domain; Fourier transform

$$\frac{\partial T(\mathbf{r},t)}{\partial t} - \kappa\nabla^2 T(\mathbf{r},t) = 0$$

$$T(\mathbf{r},0) = f(\mathbf{r})$$

$$\text{Let}: \quad \tilde{T}(\mathbf{q},t) = \int d^3r e^{-i\mathbf{q}\cdot\mathbf{r}} T(\mathbf{r},t)$$

$$\tilde{f}(\mathbf{q}) = \int d^3r e^{-i\mathbf{q}\cdot\mathbf{r}} f(\mathbf{r})$$

$$\Rightarrow \tilde{T}(\mathbf{q},0) = \tilde{f}(\mathbf{q})$$

$$\Rightarrow \frac{\partial \tilde{T}(\mathbf{q},t)}{\partial t} = -\kappa q^2 \tilde{T}(\mathbf{q},t)$$

$$\tilde{T}(\mathbf{q},t) = \tilde{T}(\mathbf{q},0) e^{-\kappa q^2 t}$$

Initial value problem in an infinite domain; Fourier transform

$$\tilde{T}(\mathbf{q},t) = \int d^3r e^{-i\mathbf{q}\cdot\mathbf{r}} T(\mathbf{r},t) \qquad \Rightarrow T(\mathbf{r},t) = \frac{1}{(2\pi)^3}\int d^3q e^{i\mathbf{q}\cdot\mathbf{r}} \tilde{T}(\mathbf{q},t)$$

$$\tilde{T}(\mathbf{q},t) = \tilde{T}(\mathbf{q},0) e^{-\kappa q^2 t}$$

$$T(\mathbf{r},t) = \frac{1}{(2\pi)^3}\int d^3q e^{i\mathbf{q}\cdot\mathbf{r}} \tilde{T}(\mathbf{q},0) e^{-\kappa q^2 t}$$

$$\tilde{T}(\mathbf{q},0) = \tilde{f}(\mathbf{q}) = \int d^3r e^{-i\mathbf{q}\cdot\mathbf{r}} f(\mathbf{r})$$

$$T(\mathbf{r},t) = \int d^3r' G(\mathbf{r}-\mathbf{r}',t) T(\mathbf{r}',0)$$

$$\text{with} \quad G(\mathbf{r}-\mathbf{r}',t) \equiv \frac{1}{(2\pi)^3}\int d^3q e^{i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')} e^{-\kappa q^2 t}$$

Initial value problem in an infinite domain; Fourier transform

$$T(\mathbf{r},t) = \int d^3r' G(\mathbf{r}-\mathbf{r}',t) T(\mathbf{r}',0)$$

$$\text{with} \quad G(\mathbf{r}-\mathbf{r}',t) \equiv \frac{1}{(2\pi)^3}\int d^3q e^{i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')} e^{-\kappa q^2 t}$$

$$G(\mathbf{r}-\mathbf{r}',t) = \frac{1}{(4\pi\kappa t)^{3/2}} e^{-|\mathbf{r}-\mathbf{r}'|^2/(4\kappa t)}$$

Heat equation in half-space

$$\frac{\partial T(\mathbf{r},t)}{\partial t} - \kappa \nabla^2 T(\mathbf{r},t) = 0$$

$T(\mathbf{r},t) \Rightarrow T(z,t)$ with initial and boundary values:

$T(z,t) \equiv 0$ for $z < 0$

$T(z,0) = 0$ for $z > 0$

$T(0,t) = T_0$ for $t \geq 0$

Solution: $T = T_0 \operatorname{erfc}\left(\frac{z}{2\sqrt{\kappa t}}\right)$

where $\operatorname{erfc}(x) \equiv \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2} du$

Heat equation in half-space -- continued

$$\frac{\partial T(z,t)}{\partial t} - \kappa \frac{\partial^2 T(z,t)}{\partial z^2} = 0$$

Solution: $T = T_0 \operatorname{erfc}\left(\frac{z}{2\sqrt{\kappa t}}\right)$

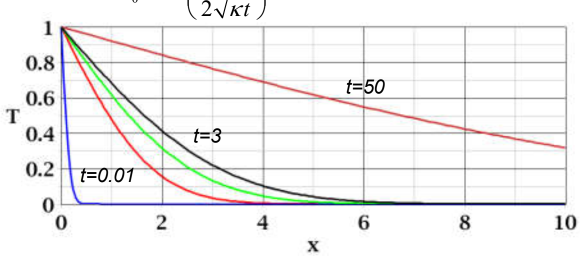
where $\operatorname{erfc}(x) \equiv \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2} du$

Note that $\frac{d \operatorname{erfc}(x)}{dx} = \frac{d}{dx} \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2} du = -\frac{2}{\sqrt{\pi}} e^{-x^2}$

$$\frac{\partial}{\partial t} \operatorname{erfc}\left(\frac{z}{2\sqrt{\kappa t}}\right) = \frac{2}{\sqrt{\pi}} e^{-\left(z^2/(4\kappa t)\right)} \left(\frac{z}{4\sqrt{\kappa t^3}}\right)$$

$$\frac{\partial^2}{\partial z^2} \operatorname{erfc}\left(\frac{z}{2\sqrt{\kappa t}}\right) = \frac{2}{\sqrt{\pi}} e^{-\left(z^2/(4\kappa t)\right)} \left(\frac{z}{4\kappa\sqrt{\kappa t^3}}\right)$$

$T = T_0 \operatorname{erfc}\left(\frac{z}{2\sqrt{\kappa t}}\right)$



Temperature profile