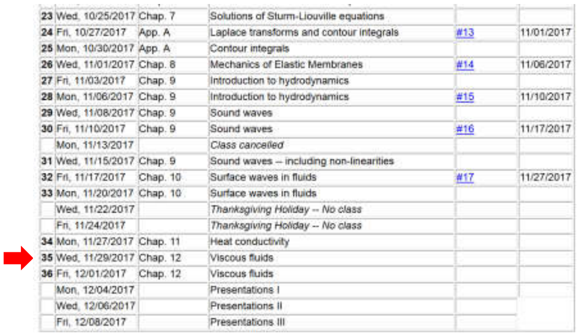
PHY 711 Classical Mechanics and Mathematical Methods
9-9:50 AM MWF Olin 107

Plan for Lecture 35

Viscous fluids – Chap. 12 in F & W

1. Viscous stress tensor
2. Navier-Stokes equation
3. Example for incompressible fluid



23	Wed, 10/25/2017	Chap. 7	Solutions of Sturm-Liouville equations		
24	Fri, 10/27/2017	App. A	Laplace transforms and contour integrals	#13	11/01/2017
25	Mon, 10/30/2017	App. A	Contour integrals		
26	Wed, 11/01/2017	Chap. 8	Mechanics of Elastic Membranes	#14	11/06/2017
27	Fri, 11/03/2017	Chap. 9	Introduction to hydrodynamics		
28	Mon, 11/06/2017	Chap. 9	Introduction to hydrodynamics	#15	11/10/2017
29	Wed, 11/08/2017	Chap. 9	Sound waves		
30	Fri, 11/10/2017	Chap. 9	Sound waves	#16	11/17/2017
	Mon, 11/13/2017		*Class cancelled*		
31	Wed, 11/15/2017	Chap. 9	Sound waves -- including non-linearities		
32	Fri, 11/17/2017	Chap. 10	Surface waves in fluids	#17	11/27/2017
33	Mon, 11/20/2017	Chap. 10	Surface waves in fluids		
	Wed, 11/22/2017		*Thanksgiving Holiday -- No class*		
	Fri, 11/24/2017		*Thanksgiving Holiday -- No class*		
34	Mon, 11/27/2017	Chap. 11	Heat conductivity		
35	Wed, 11/29/2017	Chap. 12	Viscous fluids		
36	Fri, 12/01/2017	Chap. 12	Viscous fluids		
	Mon, 12/04/2017		Presentations I		
	Wed, 12/06/2017		Presentations II		
	Fri, 12/08/2017		Presentations III		

Events

Career Event: Nov. 28 at 12:00 pm
WFU Physics Career Advising Event SPEAKER: Jeremy W. Ward, Ph.D. Airman Performance & Aeromedicine Applications Soft Matter Materials Branch Materials and Manufacturing Directorate Air Force Research Laboratory Wright-Patterson Air Force Base ...

Colloquium: Nov. 29, 2017 at 4 PM
WFU Physics Colloquium TITLE: "Kinetic Luminosity of Quasar Outflows and its Implications to Galaxy Formation" SPEAKER:Professor Nahum Arav Department of Physics Virginia Tech Blacksburg, VA TIME: Wed. Nov. 29, 2017 ...

Equations for motion of non-viscous fluid

Newton-Euler equation of motion:

$$\rho\frac{\partial \mathbf{v}}{\partial t}+\rho(\mathbf{v}\cdot\nabla)\mathbf{v}=\rho\mathbf{f}_{applied}-\nabla p$$

Continuity equation:

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\mathbf{v})=0 \qquad \Rightarrow \mathbf{v}\left(\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\mathbf{v})\right)=0$$

Add two equations:

$$\underbrace{\rho\frac{\partial \mathbf{v}}{\partial t}+\frac{\partial\rho}{\partial t}\mathbf{v}}_{\frac{\partial(\rho\mathbf{v})}{\partial t}}+\underbrace{\rho(\mathbf{v}\cdot\nabla)\mathbf{v}+\mathbf{v}\nabla\cdot(\rho\mathbf{v})}_{\sum_{j=1}^{3}\frac{\partial(\rho v_j\mathbf{v})}{\partial x_j}}=\rho\mathbf{f}_{applied}-\nabla p$$

Equations for motion of non-viscous fluid -- continued

Newton-Euler equation in terms of momentum:

$$\frac{\partial(\rho\mathbf{v})}{\partial t}+\sum_{j=1}^{3}\frac{\partial(\rho v_j\mathbf{v})}{\partial x_j}=\rho\mathbf{f}_{applied}-\nabla p$$

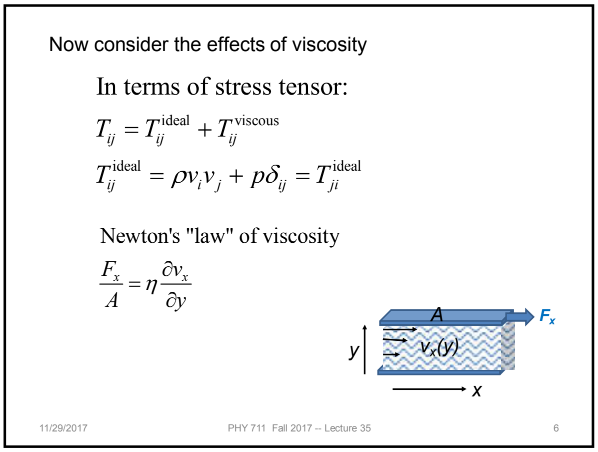
$$\frac{\partial(\rho\mathbf{v})}{\partial t}+\sum_{j=1}^{3}\frac{\partial(\rho v_j\mathbf{v})}{\partial x_j}+\nabla p=\rho\mathbf{f}_{applied}$$

Fluid momentum: $\rho\mathbf{v}$

Stress tensor: $T_{ij}\equiv\rho v_i v_j+p\delta_{ij}$

i^{th} component of Newton-Euler equation:

$$\frac{\partial(\rho v_i)}{\partial t}+\sum_{i=1}^{3}\frac{\partial T_{ij}}{\partial x_j}=\rho f_i$$



Now consider the effects of viscosity

In terms of stress tensor:

$$T_{ij}=T_{ij}^{\text{ideal}}+T_{ij}^{\text{viscous}}$$

$$T_{ij}^{\text{ideal}}=\rho v_i v_j+p\delta_{ij}=T_{ji}^{\text{ideal}}$$

Newton's "law" of viscosity

$$\frac{F_x}{A}=\eta\frac{\partial v_x}{\partial y}$$

Effects of viscosity

Argue that viscosity is due to shear forces in a fluid of the form:

$$\frac{F_{drag}}{A} = \eta \frac{\partial v_x}{\partial y}$$

Formulate viscosity stress tensor with traceless and diagonal terms:

$$T_{kl}^{\text{viscous}} = -\eta\left(\frac{\partial v_k}{\partial x_l} + \frac{\partial v_l}{\partial x_k} - \frac{2}{3}\delta_{kl}\left(\nabla\cdot\mathbf{v}\right)\right) - \zeta\delta_{kl}\left(\nabla\cdot\mathbf{v}\right)$$

viscosity

bulk viscosity

Total stress tensor: $T_{kl} = T_{kl}^{\text{ideal}} + T_{kl}^{\text{viscous}}$

$$T_{kl}^{\text{ideal}} = \rho v_k v_l + p\delta_{kl}$$

$$T_{kl}^{\text{viscous}} = -\eta\left(\frac{\partial v_k}{\partial x_l} + \frac{\partial v_l}{\partial x_k} - \frac{2}{3}\delta_{kl}\left(\nabla\cdot\mathbf{v}\right)\right) - \zeta\delta_{kl}\left(\nabla\cdot\mathbf{v}\right)$$

Effects of viscosity -- continued

Incorporating generalized stress tensor into Newton-Euler equations

$$\frac{\partial\left(\rho v_i\right)}{\partial t} + \sum_{i=1}^{3}\frac{\partial T_{ij}}{\partial x_j} = \rho f_i$$

$$\frac{\partial \rho v_i}{\partial t} + \sum_{j=1}^{3}\frac{\partial \rho v_i v_j}{\partial x_j} = \rho f_i - \frac{\partial p}{\partial x_i} + \eta\sum_{j=1}^{3}\frac{\partial^2 v_i}{\partial x_j^2} + \left(\zeta + \frac{1}{3}\eta\right)\sum_{j=1}^{3}\frac{\partial^2 v_j}{\partial x_i \partial x_j}$$

Continuity condition

$$\frac{\partial \rho}{\partial t} + \sum_{j=1}^{3}\frac{\partial \rho v_j}{\partial x_j} = 0$$

Vector form (Navier-Stokes equation)

$$\frac{\partial \mathbf{v}}{\partial t} + \left(\mathbf{v}\cdot\nabla\right)\mathbf{v} = \mathbf{f} - \frac{1}{\rho}\nabla p + \frac{\eta}{\rho}\nabla^2\mathbf{v} + \frac{1}{\rho}\left(\zeta + \frac{1}{3}\eta\right)\nabla\left(\nabla\cdot\mathbf{v}\right)$$

Continuity condition

$$\frac{\partial \rho}{\partial t} + \nabla\cdot\left(\rho\mathbf{v}\right) = 0$$

Newton-Euler equations for viscous fluids

Navier-Stokes equation

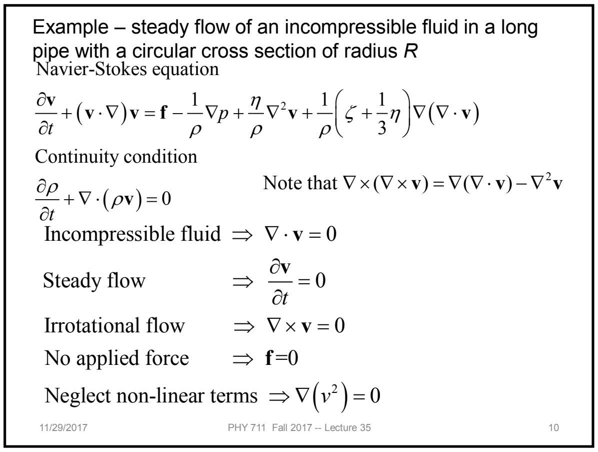
$$\frac{\partial \mathbf{v}}{\partial t} + \left(\mathbf{v}\cdot\nabla\right)\mathbf{v} = \mathbf{f} - \frac{1}{\rho}\nabla p + \frac{\eta}{\rho}\nabla^2\mathbf{v} + \frac{1}{\rho}\left(\zeta + \frac{1}{3}\eta\right)\nabla\left(\nabla\cdot\mathbf{v}\right)$$

Continuity condition

$$\frac{\partial \rho}{\partial t} + \nabla\cdot\left(\rho\mathbf{v}\right) = 0$$

Typical viscosities at 20° C and 1 atm:

Fluid	η/ρ (m^2/s)	η (Pa s)
Water	1.00 x 10^{-6}	1 x 10^{-3}
Air	14.9 x 10^{-6}	0.018 x 10^{-3}
Ethyl alcohol	1.52 x 10^{-6}	1.2 x 10^{-3}
Glycerine	1183 x 10^{-6}	1490 x 10^{-3}



Example – steady flow of an incompressible fluid in a long pipe with a circular cross section of radius *R*

Navier-Stokes equation

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{f} - \frac{1}{\rho}\nabla p + \frac{\eta}{\rho}\nabla^2\mathbf{v} + \frac{1}{\rho}\left(\zeta + \frac{1}{3}\eta\right)\nabla(\nabla\cdot\mathbf{v})$$

Continuity condition

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0$$

Note that $\nabla\times(\nabla\times\mathbf{v}) = \nabla(\nabla\cdot\mathbf{v}) - \nabla^2\mathbf{v}$

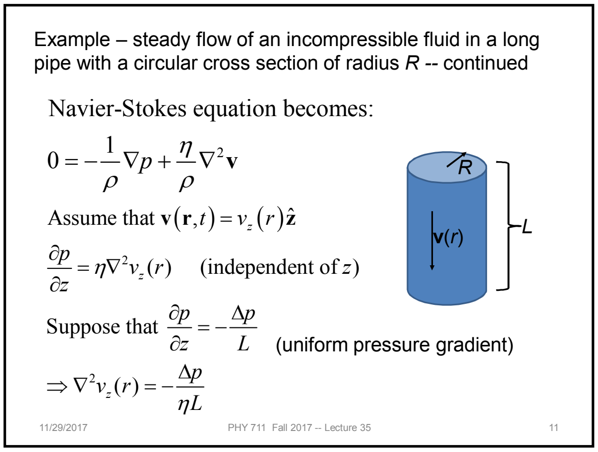
Incompressible fluid $\Rightarrow \nabla\cdot\mathbf{v} = 0$

Steady flow $\Rightarrow \frac{\partial \mathbf{v}}{\partial t} = 0$

Irrotational flow $\Rightarrow \nabla\times\mathbf{v} = 0$

No applied force $\Rightarrow \mathbf{f}=0$

Neglect non-linear terms $\Rightarrow \nabla\left(v^2\right) = 0$



Example – steady flow of an incompressible fluid in a long pipe with a circular cross section of radius *R* -- continued

Navier-Stokes equation becomes:

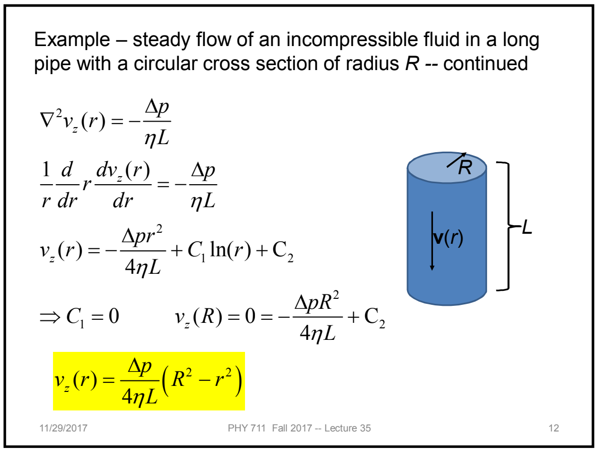
$$0 = -\frac{1}{\rho}\nabla p + \frac{\eta}{\rho}\nabla^2\mathbf{v}$$

Assume that $\mathbf{v}(\mathbf{r},t) = v_z(r)\hat{\mathbf{z}}$

$$\frac{\partial p}{\partial z} = \eta\nabla^2 v_z(r) \quad \text{(independent of } z\text{)}$$

Suppose that $\frac{\partial p}{\partial z} = -\frac{\Delta p}{L}$ (uniform pressure gradient)

$$\Rightarrow \nabla^2 v_z(r) = -\frac{\Delta p}{\eta L}$$



Example – steady flow of an incompressible fluid in a long pipe with a circular cross section of radius *R* -- continued

$$\nabla^2 v_z(r) = -\frac{\Delta p}{\eta L}$$

$$\frac{1}{r}\frac{d}{dr}r\frac{dv_z(r)}{dr} = -\frac{\Delta p}{\eta L}$$

$$v_z(r) = -\frac{\Delta p r^2}{4\eta L} + C_1\ln(r) + C_2$$

$$\Rightarrow C_1 = 0 \qquad v_z(R) = 0 = -\frac{\Delta p R^2}{4\eta L} + C_2$$

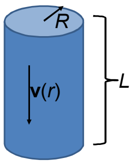
$$v_z(r) = \frac{\Delta p}{4\eta L}\left(R^2 - r^2\right)$$

Example – steady flow of an incompressible fluid in a long pipe with a circular cross section of radius *R* -- continued

$$v_z(r) = \frac{\Delta p}{4\eta L}\left(R^2 - r^2\right)$$

Mass flow rate through the pipe:

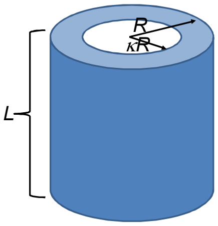
$$\frac{dM}{dt} = 2\pi\rho\int_0^R r dr v_z(r) = \frac{\Delta p \rho \pi R^4}{8\eta L}$$



Poiseuille formula;

➔Method for measuring η

Example – steady flow of an incompressible fluid in a long tube with a circular cross section of outer radius *R* and inner radius *κR*



$$\nabla^2 v_z(r) = -\frac{\Delta p}{\eta L}$$

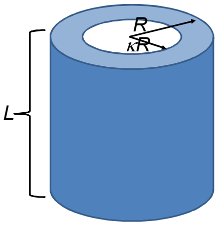
$$\frac{1}{r}\frac{d}{dr} r \frac{dv_z(r)}{dr} = -\frac{\Delta p}{\eta L}$$

$$v_z(r) = -\frac{\Delta p r^2}{4\eta L} + C_1 \ln(r) + C_2$$

$$v_z(R) = 0 = -\frac{\Delta p R^2}{4\eta L} + C_1 \ln(R) + C_2$$

$$v_z(\kappa R) = 0 = -\frac{\Delta p \kappa^2 R^2}{4\eta L} + C_1 \ln(\kappa R) + C_2$$

Example – steady flow of an incompressible fluid in a long tube with a circular cross section of outer radius *R* and inner radius *κR* -- continued



Solving for C_1 and C_2 :

$$v_z(r) = \frac{\Delta p R^2}{4\eta L}\left(1 - \left(\frac{r}{R}\right)^2 - \frac{1-\kappa^2}{\ln\kappa}\ln\left(\frac{r}{R}\right)\right)$$

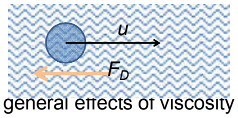
Mass flow rate through the pipe:

$$\frac{dM}{dt} = 2\pi\rho\int_{\kappa R}^R r dr v_z(r) = \frac{\Delta p \rho \pi R^4}{8\eta L}\left(1 - \kappa^4 + \frac{\left(1-\kappa^2\right)^2}{\ln\kappa}\right)$$

More discussion of viscous effects in incompressible fluids

Stokes' analysis of viscous drag on a sphere of radius R moving at speed u in medium with viscosity η:

$$F_D = -\eta(6\pi R u)$$



Plan:

1. Consider the general effects of viscosity on fluid equations
2. Consider the solution to the linearized equations for the case of steady-state flow of a sphere of radius R
3. Infer the drag force needed to maintain the steady-state flow

Newton-Euler equation for incompressible fluid, modified by viscous contribution (Navier-Stokes equation):

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{f}_{applied} - \frac{\nabla p}{\rho} + \underbrace{\frac{\eta}{\rho}}_{\upsilon}\nabla^2\mathbf{v}$$

Kinematic viscosity

Typical kinematic viscosities at 20° C and 1 atm:

Fluid	ν (m²/s)
Water	1.00 x 10^{-6}
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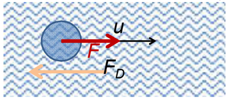
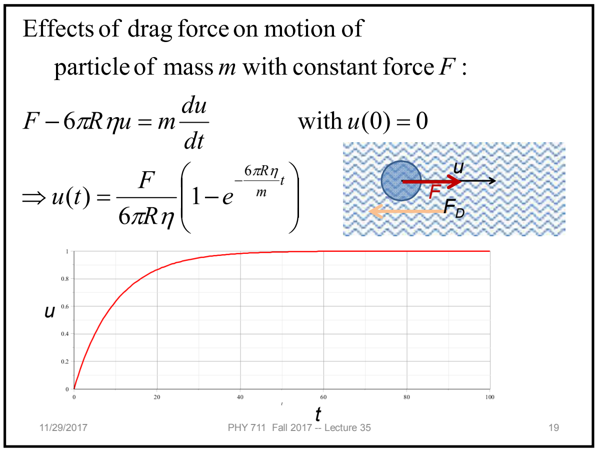


Effects of drag force on motion of particle of mass m with constant force F:

$$F - 6\pi R\eta u = m\frac{du}{dt} \qquad \text{with } u(0) = 0$$

$$\Rightarrow u(t) = \frac{F}{6\pi R\eta}\left(1 - e^{-\frac{6\pi R\eta}{m}t}\right)$$



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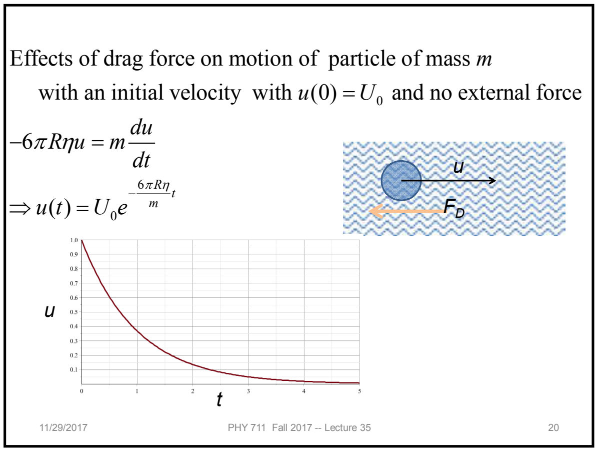
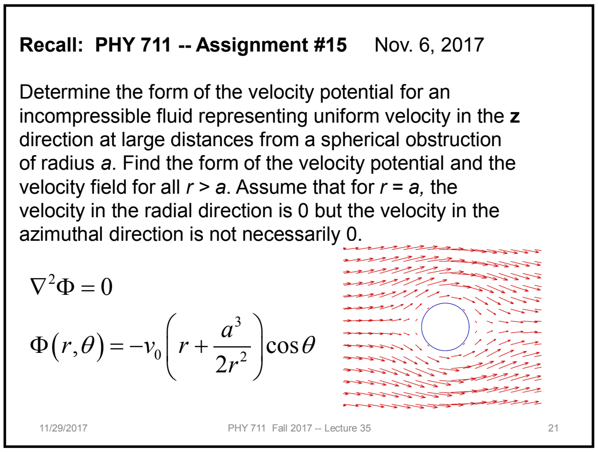


Effects of drag force on motion of particle of mass m with an initial velocity with $u(0) = U_0$ and no external force

$$-6\pi R\eta u = m\frac{du}{dt}$$

$$\Rightarrow u(t) = U_0 e^{-\frac{6\pi R\eta}{m}t}$$



Recall: PHY 711 -- Assignment #15 Nov. 6, 2017

Determine the form of the velocity potential for an incompressible fluid representing uniform velocity in the **z** direction at large distances from a spherical obstruction of radius *a*. Find the form of the velocity potential and the velocity field for all *r* > *a*. Assume that for *r* = *a*, the velocity in the radial direction is 0 but the velocity in the azimuthal direction is not necessarily 0.

$$\nabla^2\Phi = 0$$

$$\Phi(r,\theta) = -v_0\left(r + \frac{a^3}{2r^2}\right)\cos\theta$$

Newton-Euler equation for incompressible fluid, modified by viscous contribution (Navier-Stokes equation):

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{f}_{applied} - \frac{\nabla p}{\rho} + \frac{\eta}{\rho}\nabla^2\mathbf{v}$$

Continuity equation: $\nabla\cdot\mathbf{v} = 0$

Assume steady state: $\Rightarrow \frac{\partial \mathbf{v}}{\partial t} = 0$

Assume non-linear effects small

Initially set $\mathbf{f}_{applied} = 0;$

$$\Rightarrow \nabla p = \eta\nabla^2\mathbf{v}$$

$$\nabla p = \eta\nabla^2\mathbf{v}$$

Take curl of both sides of equation:

$$\nabla\times(\nabla p) = 0 = \eta\nabla^2(\nabla\times\mathbf{v})$$

Assume (with a little insight from Landau):

$$\mathbf{v} = \nabla\times(\nabla\times f(r)\mathbf{u}) + \mathbf{u}$$

where $f(r)\xrightarrow[r\to\infty]{}0$

Note that:

$$\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$$

Digression

Some comment on assumption: $\mathbf{v} = \nabla\times(\nabla\times f(r)\mathbf{u}) + \mathbf{u}$

$$\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$$

Here $\mathbf{A} = f(r)\mathbf{u}$

$$\nabla\times\mathbf{v} = \nabla\times(\nabla\times(\nabla\times\mathbf{A})) = -\nabla\times(\nabla^2\mathbf{A})$$

Also note: $\nabla p = \eta\nabla^2\mathbf{v}$

$\Rightarrow \nabla\times\nabla p = \nabla\times\eta\nabla^2\mathbf{v}$ or $\nabla^2(\nabla\times\mathbf{v}) = 0$

$$\mathbf{v} = \nabla\times\left(\nabla\times f(r)\mathbf{u}\right)+\mathbf{u}$$
$$\mathbf{u} = u\hat{\mathbf{z}}$$
$$\nabla\times\left(\nabla\times f(r)\hat{\mathbf{z}}\right)=\nabla\left(\nabla\cdot f(r)\hat{\mathbf{z}}\right)-\nabla^2 f(r)\hat{\mathbf{z}}$$
$$\nabla\times\mathbf{v}=0 \qquad \Rightarrow\nabla^2\left(\nabla\times\mathbf{v}\right)=0$$
$$\nabla^4\left(\nabla\times f(r)\hat{\mathbf{z}}\right)=0 \quad \Rightarrow\nabla^4\left(\nabla f(r)\times\hat{\mathbf{z}}\right)=0 \quad \Rightarrow\nabla^4 f(r)=0$$
$$f(r)=C_1r^2+C_2r+C_3+\frac{C_4}{r}$$
$$v_r=u\cos\theta\left(1-\frac{2}{r}\frac{df}{dr}\right)=u\cos\theta\left(1-4C_1-\frac{2C_2}{r}-\frac{2C_4}{r^3}\right)$$
$$v_\theta=-u\sin\theta\left(1-\frac{d^2f}{dr^2}-\frac{1}{r}\frac{df}{dr}\right)=-u\sin\theta\left(1-4C_1-\frac{C_2}{r}+\frac{C_4}{r^3}\right)$$

Some details:

$$\nabla^4 f(r)=0 \qquad \Rightarrow\left(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}\right)^2 f(r)=0$$
$$f(r)=C_1r^2+C_2r+C_3+\frac{C_4}{r}$$
$$\mathbf{v}=u\Big(\nabla\times\left(\nabla\times f\left(r\right)\hat{\mathbf{z}}\right)+\hat{\mathbf{z}}\Big)$$
$$=u\Big(\nabla\Big(\nabla\cdot\left(f\left(r\right)\hat{\mathbf{z}}\right)\Big)-\nabla^2 f(r)\hat{\mathbf{z}}+\hat{\mathbf{z}}\Big)$$

Note that: $\hat{\mathbf{z}}=\cos\theta\hat{\mathbf{r}}-\sin\theta\hat{\boldsymbol{\theta}}$

$$\mathbf{v}=u\left(\nabla\left(\frac{df}{dr}\cos\theta\right)-\left(\nabla^2\left(f(r)\right)-1\right)\left(\cos\theta\hat{\mathbf{r}}-\sin\theta\hat{\boldsymbol{\theta}}\right)\right)$$

$$v_r=u\cos\theta\left(1-\frac{2}{r}\frac{df}{dr}\right)=u\cos\theta\left(1-4C_1-\frac{2C_2}{r}-\frac{2C_4}{r^3}\right)$$
$$v_\theta=-u\sin\theta\left(1-\frac{d^2f}{dr^2}-\frac{1}{r}\frac{df}{dr}\right)=-u\sin\theta\left(1-4C_1-\frac{C_2}{r}+\frac{C_4}{r^3}\right)$$

To satisfy $\mathbf{v}(r\to\infty)=\mathbf{u}$: $\qquad\Rightarrow C_1=0$

To satisfy $\mathbf{v}(R)=0$ solve for C_2, C_4

$$v_r=u\cos\theta\left(1-\frac{3R}{2r}+\frac{R^3}{2r^3}\right)$$
$$v_\theta=-u\sin\theta\left(1-\frac{3R}{4r}-\frac{R^3}{4r^3}\right)$$

$$v_r = u\cos\theta\left(1-\frac{3R}{2r}+\frac{R^3}{2r^3}\right)$$

$$v_\theta = -u\sin\theta\left(1-\frac{3R}{4r}-\frac{R^3}{4r^3}\right)$$

Determining pressure:

$$\nabla p = \eta\nabla^2\mathbf{v} = -\eta\nabla\left(u\cos\theta\left(\frac{3R}{2r^2}\right)\right)$$

$$\Rightarrow p = p_0 - \eta u\cos\theta\left(\frac{3R}{2r^2}\right)$$

$$p = p_0 - \eta u\cos\theta\left(\frac{3R}{2r^2}\right)$$

Corresponds to:

$$F_D\cos\theta = \left(p(R)-p_0\right)4\pi R^2$$

$$\Rightarrow \mathrm{F}_D = -\eta u\left(6\pi R\right)$$